Contribution of Mihailo Petrović to the Development of Interval Arithmetic and Computation with Numerical Intervals*

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Abstract

In this short historical note, we show that the outstanding Serbian mathematician Mihailo Petrović (1868–1943) was one of the pioneers in the foundation of interval arithmetic and calculation with numerical intervals. A short review of results included in his book *Calculations with numerical intervals*, published in 1932 in Serbian, is presented.

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Calculation with intervals has expanded at the beginning of sixties of the twentieth century with the rapid progress of digital computers. The demands of the computer age closely connected to *"the effect of rounding errors and propagation error due to uncertain initial data or uncertain values of parameters in mathematical models"* (R. E. Moore [1]), which dictated the need for a structure referred to as *interval arithmetics*. The beginnings of interval arithmetic date from 1958-59 when T. Sunaga [2] and R. E. Moore [3] used intervals for the first time to be able to bound the round-off error in computing by means of digital computers. However, perhaps R. C. Young, cited by Moore (see [4]), should be treated as the forerunner of interval arithmetic. She has introduced in [5] the operations, not with intervals but with arbitrary sets of real numbers (many-valued quantities).

In this short note, we reveal a little known fact that, even before R. C. Young, a Serbian mathematician Mihailo Petrović held a course *"Calculations with Numerical Intervals"* at the Belgrade University, then in Yugoslavia. This author even published a textbook [6] (Belgrade, 1932) under the same title. The aim of this paper is to

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present M. Petrović as one of the pioneers of interval mathematics, considering that the scientific public is not well acquainted with his work in the field of interval arithmetic.

Mihailo Petrović (1868–1943), born in Belgrade, was one of the greatest Serbian mathematicians. He had studied at Faculté des sciences in Paris, where he received his Ph. D. titled: *Sur les zéros et les infinis des intégrales des équations différentielles algebrique* (1894) in the presence of Ph. D. committee of professors: Hermite (president), Picard, Painlevé (examiners). For a half of century, from 1894 till 1943, Petrović was indefatigably working as a professor of High School in Belgrade, creating and training the scientific and teaching staff. In that period he had published about 235 papers and 12 books. Almost all doctoral dissertations in Serbia in the field of Mathematics between two World Wars were done under his direction. Petrović's research area of interest included arithmetics, inequalities, polynomials, complex analysis, differential equations, integral calculus, and general phenomenology. He was a member of many scientific societies and many Academies of Sciences. It is interesting to note that professor Petrović was also a world traveler who visited many countries over the world and even stayed in the Northern and Southern poles. In addition to these adventurous trips, he was a passionate fisherman and musician (violinist, a member of musical ensemble) – indeed, a versatile and unusual person.

M. Petrović was one of the first who had noticed the importance and significance of interval calculus. In connection with that, we cite some arguments given by R. E. Moore thirty years later in [4]:

> *"In practice (except some special cases), a real quantity is not determined as an exact number or mathematical point, but as a numerical interval; a point is determined as a segment of a straight line or curve; a line in the plene as a track of finite width; a line in space as a stick of finite thickness, e.t.c. Thus, the elements of mathematics in practice are not the same as the elements of abstract mathematics. However, in abstract mathematics the problem of special kind exists, where the unknown quantities naturally appear as numerical intervals: in the case when conditions of problem do not require the exact value of the unknown quantity or exact determination is not possible due to unsurmountable difficulties; further, when it is sufficient to find the interval containing the unknown quantity and, on the basis of that, its exact value can be determined; then, when the solutions of problems are irrational numbers (e.g. the roots of equation* $x^2 - 2 = 0$*), etc. ..."*

M. Petrović applied intervals in various domains of mathematics, many of which mentioned in the above Moore's citation. However, he did not work in the same way as we do nowadays, which can be observed by reading his book *"Calculations with numerical intervals"*, the subject of our short review. This book consists of three chapters given below together with titles of sections (written in italic):

Ch. 1: Numerical intervals in elementary calculus

> *Numerical intervals as mathematical elements, Representations of numerical intervals, Linear represents of numerical intervals, Transformation of represents of numerical intervals, Functions of numerical intervals, Functions of several numerical intervals, System of functions of numerical intervals, Constant and variable numerical intervals, Numerical intervals in Arithmetics and Algebra, Numerical intervals in the Theory of errors, Numerical intervals in Geometry, Complete and incomplete functional dependency*

Ch. 2: Numerical intervals in infinitesimal calculus

> *Differentiation and integration of numerical intervals, Definite integrals as numerical intervals, Standard Mean value theorem for integrals, Second Mean value theorem for integrals, Curvilinear integrals as numerical intervals, Surface in space as numerical intervals, Various classes of definite integrals as numerical intervals*

Ch. 3: Numerical intervals for integrals of differential equations

> *Comparative study of differential equations, First method, Second method, Qualitative fists integrals of differential equations, Numerical intervals determined by qualitative first integrals, Differential equations of the first and second order with oscillatory integrals, Intervals for integrals of systems of simultaneous equations, Intervals for integrals of partial differential equations*

Chapters 2 and 3 in Petrović's book [6] mostly contain the estimations of values of definite integrals when the integrand is between two given curves on the segment of integration. In order to determine bounds of results of considered problems in the topic of definite integrals or solutions of differential equations, Petrović stated efficient and applicable estimating procedures based on various types of inequalities, most frequently followed by illustrative examples and apposite discussions and comments. Although general methods for bounding computed results or solutions of equations (of various type) are essentially different from modern interval algorithms, Petrović original ideas have been a fruitful base for estimating the bounds of results of numerical computations in several mathematics fields.

The results that can be included in contemporary interval mathematics are given in Chapter 1. In what follows we give some representations of real intervals that were introduced by Petrović.

Given a real interval $[a, b]$, a real function f of a real parameter $\boldsymbol{l} \in [\boldsymbol{l}_1, \boldsymbol{l}_2]$ can be formed such that

1) $f(\boldsymbol{l}_1) = a,\ f(\boldsymbol{l}_2) = b$;
2) $f(\boldsymbol{l}) \in [a, b],\ \boldsymbol{l} \in [\boldsymbol{l}_1, \boldsymbol{l}_2]$.

The function $\boldsymbol{l} \to f(\boldsymbol{l})$ is called *the computational representative* of the interval $[a, b]$, while the interval $[\boldsymbol{l}_1, \boldsymbol{l}_2]$ is named *the parameter interval* (followed Petrović's terminology). For example, if

$$\boldsymbol{a} = \frac{\boldsymbol{l} - \boldsymbol{l}_2}{\boldsymbol{l}_1 - \boldsymbol{l}_2},\ \beta = \frac{\boldsymbol{l} - \boldsymbol{l}_1}{\boldsymbol{l}_2 - \boldsymbol{l}_1},$$

then for the computational representative of the interval $[a, b]$ the following functions can be taken:

$$f(\boldsymbol{l}) = \boldsymbol{a}a + \beta b,\ f(\boldsymbol{l}) = (\boldsymbol{a}a^m + \beta b^m)^{1/m}\ (m \in N),\ f(\boldsymbol{l}) = a^{\boldsymbol{a}} b^{\beta}.$$

Among all possible forms of the function f the simplest is linear function of the parameter $\boldsymbol{l}$, i.e.

$$f(\boldsymbol{l}) = u + \boldsymbol{l}v,$$

where u and v are real numbers independent on $\boldsymbol{l}$. These values are determined from the conditions

$$u + \boldsymbol{l}_1 v = a,\ u + \boldsymbol{l}_2 v = b$$

for a given interval $[a,b]$. Hence,

$$u = \frac{\boldsymbol{l}_2}{\boldsymbol{l}_2 - \boldsymbol{l}_1} a - \frac{\boldsymbol{l}_1}{\boldsymbol{l}_2 - \boldsymbol{l}_1} b, \ v = \frac{1}{\boldsymbol{l}_2 - \boldsymbol{l}_1} b - \frac{1}{\boldsymbol{l}_2 - \boldsymbol{l}_1} a.$$

The following two simple cases are interesting:

1. $\boldsymbol{l}_1 = -1, \boldsymbol{l}_2 = 1$; the computational representative of the interval $[a,b]$ is

$$f(\boldsymbol{l}) = \frac{b+a}{2} + \boldsymbol{l}\frac{b-a}{2}, \tag{1}$$

   and the parameter interval is $[-1,1]$.

2. $\boldsymbol{l}_1 = 0, \boldsymbol{l}_2 = 1$; the computational representative is

$$f(\boldsymbol{l}) = a + \boldsymbol{l}(b-a), \tag{2}$$

   and the parameter interval is $[0,1]$.

According to (1) and (2) there follows

$$[a,b] = \frac{b+a}{2} + [-1,1]\frac{b-a}{2}$$

and

$$[a,b] = a + [0,1](b-a).$$

These forms of the interval $[a,b]$ are named *the symmetrical* and *asymmetrical normal form* of the given interval, respectively. In the case of the symmetrical form $u + \boldsymbol{l}v$ the term $u = (b+a)/2$ is *the midpoint* of $[a,b]$ and $v = (b-a)/2$ is *the semiwidth.* Note that T. Sunaga used the form (1) in [2].

Beside the mentioned forms for representation of real intervals, M. Petrović introduced the functions of one or several intervals, and the systems of interval functions, too. From the modern point of view, only functions of one interval are important for interval mathematics, so that we restrict our considerations to that case.

Let $X = [x_1, x_2]$ be a real interval and q a given real-valued function defined over X. Then $Z = g(X) = [z_1, z_2]$ is a real interval, too. If

$$N = \min_{x \in X} g(X), \ M = \max_{x \in X} g(X),$$

the interval Z have the symmetrical normal form

$$Z = g(X) = \frac{M+N}{2} + [-1,1]\frac{M-N}{2},$$

while the asymmetrical normal form is given by

$$Z = g(X) = N + [0,1](M-N).$$

When g is a monotonic function of x on the interval X, then

$$N = \min\{g(x_1), g(x_2)\}, \ M = \max\{g(x_1), g(x_2)\}.$$

M. Petrović represented a lot of examples of interval functions where intervals are expressed in the normal form, of which we cite the following one.

Example. Let the interval $X = [a,b]$ be given by its asymmetrical form

$$X = u + [0,1]v \quad (u = a, \ v = b - a).$$

Assume that the function $x \mapsto f(x)$ is monotonically increasing over X. Then

$$f(X) = f(u + [0,1]v) = f(u) + [0,1](f(u+v) - f(v)).$$

In the same way the monotonically decreasing function is treated.

For example, for $a > 0$ the logarithm of the interval $X = [a, b]$ is given by

$$\log X = \log(u + [0,1]v) = \log a + [0,1] \log \frac{b}{a}.$$

In conclusion we can say that, although Petrović's approach is not competitive with contemporary interval methods, it is certainly of historical interest. Moreover, Petrović's book has been of great importance since the author offered a variety of original mathematical ideas used in later works of mathematicians over the world. Among them, "Petrović's" real intervals with sliding parameter could be treated as an early idea that precedes the coming of fuzzy sets many years later.

References

[1] R. E. Moore, *Methods and Applications of Interval Analysis.* SIAM Studies in Applied Mathematics, SIAM, Philadelphia, 1979.

[2] T. Sunaga, Theory of an interval algebra and its applications to numerical analysis. *RAAG Memoirs* 2: 29–46, 1958.

[3] R. E. Moore, Automatic error analysis in digital computation, LMSD-48421, Lockeed Missiles and Space Co., Palo Alto, 1959.

[4] R. E. Moore, *Interval Analysis.* Prentice-Hall, Inc. Englewood Cliffs, New Jersey, 1966.

[5] R. C. Young, The algebra of many-valued quantities. *Math. Anal.* 104: 260–290, 1931.

[6] M. Petrović, *Računanje sa brojnim razmacima* (in Serbian, Calculations with Numerical Intervals (title translated to English). Belgrade, 1932. Second edition 1969.