Euclidean Distance Between Intervals Is the Only Representation-Invariant One*

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Abstract

An interval can be represented as a point in a plane, e.g., as a point with its endpoints as coordinates. We can thus define distance between intervals as the Euclidean distance between the corresponding points. Alternatively, we can describe an interval by its center and radius, which leads to a different definition of distance. Interestingly, these two definitions lead, in effect, to the same distance – to be more precise, these two distances differ by a multiplicative constant. In principle, we can have more general distances on the plane. In this paper, we show that only for Euclidean distance, the two representations lead to the same distance between intervals.

1 Formulation of the Problem

Need for interval uncertainty. Most information about physical quantities comes from measurements, and measurements are never absolutely exact: the actual (unknown) value x of a physical quantity is, in general, different from the measurement result $\widetilde{x}$.

In many real-life situations, the only information that we have about the measurement error $\Delta x \stackrel{\text{def}}{=} \widetilde{x} - x$ is the upper bound Δ on its absolute value: $|\Delta x| \leq \Delta$. In this case, based on the measurement result, the only information that we gain about the actual value x is that this value is somewhere in the interval $[\underline{x}, \overline{x}]$, where $\underline{x} = \widetilde{x} - \Delta$ and $\overline{x} = \widetilde{x} + \Delta$; see, e.g., [2, 4, 5, 6].

Two representations of intervals. In line with the discussion above, we have two natural representations of an interval:

- we can represent an interval by its midpoint $\widetilde{x}$ and radius (half-width) Δ;
- alternatively, we can represent an interval by its endpoints $\underline{x} = \widetilde{x} - \Delta$ and $\overline{x} = \widetilde{x} + \Delta$.

It is important to gauge the distance between intervals. In many practical situations, we are interested in the value of a quantity y which is not easy to directly measure – e.g., we are interested in future values of some quantities. To estimate these values, we:

- find easier-to-measure quantities $x_1, \ldots, x_n$ which are related to the desired quantity y by a known relation $y = f(x_1, \ldots, x_n)$,
- measure these quantities x_i, and
- use the measurement results (and the known relation) to estimate y.

As we have mentioned, often, based on the measurement results, the only information that we get about the actual (unknown) value of each quantity x_i is the interval $[\underline{x}_1, \overline{x}_i]$ that contains this value. In this case, the only information that we can gain about the desired quantity y is that it belongs to the corresponding set

$$\{f(x_1, \ldots, x_n) : x_1 \in [\underline{x}_1, \overline{x}_1], \ldots, x_n \in [\underline{x}_n, \overline{x}_n]\}. \tag{1}$$

For a continuous function $f(x_1, \ldots, x_n)$, this set is also an interval. The problem of computing this interval is known as the problem of *interval computation*; see, e.g., [2, 4, 5].

It is known that this problem is, in general, NP-hard (see, e.g., [3]), which means that – unless it turns out that P=NP – no feasible algorithm is possible that would always compute the exact endpoints of the interval (1). In situations when we cannot compute the desired interval exactly, we can only compute an *approximation* to this interval. To understand how good is this approximation, it is important to have a natural way to define the distance between two intervals.

Two natural representations lead to two natural definitions of the distance between intervals. If we represent an interval by a pair $(\widetilde{x}, \Delta)$, then it is natural to define the distance between two intervals as the Euclidean distance between the corresponding 2-D points

$$d((\widetilde{x}_1, \Delta_1), (\widetilde{x}_2, \Delta_2)) = \sqrt{(\widetilde{x}_1 - \widetilde{x}_2)^2 + (\Delta_1 - \Delta_2)^2}. \tag{2}$$

On the other hand, if we represent an interval by its endpoints $(\underline{x}, \overline{x})$, then it is natural to define the distance between two intervals as the Euclidean distance between the corresponding 2-D points:

$$D((\underline{x}_1, \overline{x}_1), (\underline{x}_2, \overline{x}_2)) = \sqrt{(\underline{x}_1 - \underline{x}_2)^2 + (\overline{x}_1 - \overline{x}_2)^2}. \tag{3}$$

The two metrics differ by a multiplicative constant. The two metrics differ by a multiplicative constant. Indeed, if we denote $a \stackrel{\text{def}}{=} \widetilde{x}_1 - \widetilde{x}_2$ and $b \stackrel{\text{def}}{=} \Delta_1 - \Delta_2$, then these distances are given by

$$d = \sqrt{a^2 + b^2} \tag{4}$$

and

$$D = \sqrt{(a - b)^2 + (a + b)^2}. \tag{5}$$

The expression under the square root is equal to

$$(a - b)^2 + (a + b)^2 = 2(a^2 + b^2), \tag{6}$$

thus $D = \sqrt{2} \cdot d$.

Instead of distances, we can consider their squares d^2 and D^2, then we have

$$D^2 = 2d^2. \tag{7}$$

Comment. In addition to the above two natural representations, it is possible to have other representations of intervals. For example, we can represent an interval $[\underline{x}, \overline{x}]$ by:

- its lower endpoint $\underline{x}$, and
- either 1) its width $\overline{x} - \underline{x} = 2\Delta$ or 2) its half-width Δ.

In these representations, the corresponding Euclidean distances

$$d_1((\underline{x}_1, \overline{x}_1), (\underline{x}_2, \overline{x}_2)) = \sqrt{(\underline{x}_1 - \underline{x}_2)^2 + (2\Delta_1 - 2\Delta_2)^2} = \sqrt{(a - b)^2 + 4b^2} =$$
$$\sqrt{a^2 - 2a \cdot b + 5b^2} \tag{8}$$

and

$$d_2((\underline{x}_1, \overline{x}_1), (\underline{x}_2, \overline{x}_2)) = \sqrt{(\underline{x}_1 - \underline{x}_2)^2 + (\Delta_1 - \Delta_2)^2} = \sqrt{(a - b)^2 + b^2} =$$
$$\sqrt{a^2 - 2a \cdot b + 2b^2} \tag{9}$$

are different from the distances d and D based on the natural representations.

Natural question. In the 2-D plane, instead of the square of the Euclidean distance $(x_1 - x_2)^2 + (y_1 - y_2)^2$, we can consider more general expressions

$$f(x_1 - x_2) + f(y_1 - y_2), \tag{10}$$

for some even function $f(X)$ which is increasing for $x \geq 0$. For example, for $f(X) = |x|^p$, we get ℓ_p-distance.

A natural question is: for which functions $f(X)$, the values of corresponding f-distance (10) corresponding to two representations of an interval always differ by a multiplicative constant?

What we do in this paper. In this paper, we prove that the only functions $f(X)$ with this property are functions of the type $f(X) = c \cdot x^2$ corresponding to the usual Euclidean distance.

2 Main Result

Discussion. In terms of the above-defined differences a and b, the desired property has the following form:

$$f(a + b) + f(a - b) = C \cdot (f(a) + f(b)). \tag{11}$$

Theorem. *The following two conditions are equivalent:*

- $f(X)$ *is an even function which is strictly increasing for* $x \geq 0$ *and for which there exists a constant C for which (11) holds for all a and b;*

- $f(X) = c \cdot x^2$ *for some constant $c > 0$.*

Comment. Our functional equation is somewhat similar to Cauchy's functional equation

$$f(a + b) = f(a) + f(b); \tag{12}$$

see, e.g., [1] – and our proof is motivated by Cauchy's proof that the only continuous solutions to the functional equation (12) are linear functions $f(X) = k \cdot x$.

This analogy raises several open questions: in our result, can we replace monotonicity with continuity? do we need to require evenness? It would be interesting to find answers to these questions.

Proof.

$0°$. Clearly, the function $f(X) = c \cdot x^2$ is an even function which is strictly increasing for $x \geq 0$ and for which there exists a constant $C = 2$ for which (11) holds for all a and b.

So, to prove the proposition, it is sufficient to prove that if a function $f(X)$ is an even function which is strictly increasing for $x \geq 0$ and for which there exists a constant C for which (11) holds for all a and b, then $f(X) = c \cdot x^2$ for some $c > 0$. Let us now assume that $f(X)$ is such a function.

$1°$. Let us first prove that $C = 2$ and $f(0) = 0$.

Indeed, for $b = 0$, the formula (11) takes the form $2f(a) = C \cdot (f(a) + f(0))$, i.e., the form $(2 - C) \cdot f(a) = C \cdot f(0)$. We cannot have $C \neq 2$ since then the expression $(2 - C) \cdot f(a)$ would be either increasing or decreasing and will not be equal to a constant $C \cdot f(0)$. Thus, $C = 2$ and hence, $f(0) = 0$. For $C = 2$, the equality (11) take the form

$$f(a + b) + f(a - b) = 2(f(a) + f(b)). \tag{13}$$

$2°$. Let us prove, by induction, that for every $n \geq 1$, we have

$$f(n \cdot a) = n^2 \cdot f(a). \tag{16}$$

Indeed, for $n = 1$, this is trivially true. So, we have the induction base.

Let us now prove the induction step. Let us assume that we have proved (16) for all $n = 1, \ldots, k$, let us prove that this equality holds for $n = k + 1$ as well. For this, let is take $b = k \cdot a$. Then, the formula (13) takes the form

$$f((k + 1) \cdot a) + f((k - 1) \cdot a) = 2(f(a) + f(k \cdot a)),$$

hence

$$f((k + 1) \cdot a) = 2(f(a) + f(k \cdot a)) - f((k - 1) \cdot a). \tag{15}$$

We already know that

$$f(k \cdot a) = k^2 \cdot f(a), \tag{16}$$

and that

$$f((k - 1) \cdot a) = (k - 1)^2 \cdot f(a). \tag{17}$$

Substituting (16) and (17) into (15), we get

$$f((k+1) \cdot a) = 2(f(a) + k^2 \cdot f(a)) - (k-1)^2 \cdot f(a) = (2 + 2k^2 - (k-1)^2) \cdot f(a) =$$

$$(2 + 2k^2 - k^2 + 2k - 1) \cdot f(a) = (k^2 + 2k + 1) \cdot f(a) = (k+1)^2 \cdot f(a). \tag{18}$$

So, by induction, the formula (14) is indeed true for all n.

3°. Let us now prove that

$$f(r) = c \cdot r^2 \tag{19}$$

for all rational numbers $r = \dfrac{p}{q}$.

Indeed, for $n = q$ and $a = \dfrac{1}{q}$, the formula (14) implies that

$$f(1) = q^2 \cdot f\left(\frac{1}{q}\right), \tag{20}$$

hence

$$f\left(\frac{1}{q}\right) = c \cdot \left(\frac{1}{q}\right)^2, \tag{21}$$

where we denoted $c \stackrel{\text{def}}{=} f(1)$.

Now, for $n = p$ and $a = \dfrac{1}{q}$, the formula (14) implies that

$$f\left(\frac{p}{q}\right) = p^2 \cdot f\left(\frac{1}{q}\right). \tag{22}$$

Substituting the expression (21) for $f\left(\dfrac{1}{q}\right)$ into the formula (22), we conclude that

$$f\left(\frac{p}{q}\right) = c \cdot \left(\frac{p}{q}\right)^2. \tag{23}$$

The statement is proven.

4°. To complete the proof, we need to show that the formula $f(X) = c \cdot x^2$ holds for all real values $x \geq 0$. Indeed, for each q, each real number can be approximated, from below and from above, by fractions

$$\frac{p(q)}{q} \leq x \leq \frac{p(q) + 1}{q}, \tag{24}$$

where $p(q) \stackrel{\text{def}}{=} \lfloor q \cdot x \rfloor$. Since the function $f(X)$ is increasing for $x \geq 0$, we have

$$f\left(\frac{p(q)}{q}\right) \leq f(x) \leq f\left(\frac{p(q) + 1}{q}\right). \tag{25}$$

Due to Part 3 of this proof, we have

$$c \cdot \left(\frac{p(q)}{q}\right) \leq f(x) \leq c \cdot \left(\frac{p(q) + 1}{q}\right)^2. \tag{26}$$

In the limit $q \to \infty$, both the left- and the right-hand sides of this double inequality tend to $c \cdot x^2$, this indeed $f(X) = c \cdot x^2$ for all $x \geq 0$.

Since the function $f(X)$ is even, this equality is true for all r. The proposition is proven.

Acknowledgments

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