

# How to Extend Interval Arithmetic So That Inverse and Division Are Always Defined\*

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## Abstract

In many real-life data processing situations, we only know the values of the inputs with interval uncertainty. In such situations, it is necessary to take this interval uncertainty into account when processing data. Most existing methods for dealing with interval uncertainty are based on interval arithmetic, i.e., on the formulas that describe the range of possible values of the result of an arithmetic operation when the inputs are known with interval uncertainty. For most arithmetic operations, this range is

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also an interval, but for division, the range is sometimes a disjoint union of two semi-infinite intervals. It is therefore desirable to extend the formulas of interval arithmetic to the case when one or both inputs is such a union. The corresponding extension is described in this paper.

**Keywords:** interval uncertainty, interval arithmetic, interval division, union of two semi-infinite intervals

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## 1 Formulation of the Problem

**Need for data processing.** We want to understand the state of the world, we want to understand what will happen in the future and how to make this future better. Each state is described by the values of different quantities. Some quantities we can measure directly, others – like the distance to the Sun – we cannot measure directly. The only way to find the values of such quantities  $y$  is:

- to find easier-to-measure quantities  $x_1, \dots, x_n$  that are related to  $y$  by a known dependence  $y = f(x_1, \dots, x_n)$ ,
- to measure these quantities, and
- then to estimate  $y$  by plugging in the measurement results  $\tilde{x}_i$  into the known dependence, i.e., to compute the value  $\tilde{y} = f(\tilde{x}_1, \dots, \tilde{x}_n)$ .

An important case is when  $y$  is the future value of some quantity, then, of course, we cannot measure it now, but we can often predict it by using the current values of related quantities.

In all these cases, computing  $y = f(x_1, \dots, x_n)$  based on the known values of  $x_i$  is known as *data processing*.

**Need to take interval uncertainty into account.** The values  $\tilde{x}_i$  come from measurements. Measurements are never absolutely accurate: the measurement result  $\tilde{x}$  is, in general, different from the actual (unknown) value  $x$  of the corresponding quantity. Often, the only information that we have about the measurement error  $\Delta x \stackrel{\text{def}}{=} \tilde{x} - x$  is the upper bound  $\Delta$  on its absolute value:  $|\Delta x| \leq \Delta$ ; see, e.g., [4]. In this case, once we know the measurement result  $\tilde{x}$ , the only information that we have about the actual value  $x$  is that this value belongs to the interval  $[\underline{x}, \bar{x}]$ , where  $\underline{x} \stackrel{\text{def}}{=} \tilde{x} - \Delta$  and  $\bar{x} \stackrel{\text{def}}{=} \tilde{x} + \Delta$ .

So, for each  $i$ , we do now know the exact value  $x_i$ , we only know the interval  $X_i = [\underline{x}_i, \bar{x}_i]$  of possible values. Different combinations of values  $x_i$  from these intervals lead, in general, to different values  $y = f(x_1, \dots, x_n)$ . The only thing we can then say about  $y$  is that it belongs to the range of all such values

$$f(X_1, \dots, X_n) \stackrel{\text{def}}{=} \{f(x_1, \dots, x_n) : x_1 \in X_1, \dots, x_n \in X_n\}, \quad (1)$$

i.e., in this case,

$$f([\underline{x}_1, \bar{x}_1], \dots, [\underline{x}_n, \bar{x}_n]) \stackrel{\text{def}}{=} \{f(x_1, \dots, x_n) : x_1 \in [\underline{x}_1, \bar{x}_1], \dots, x_n \in [\underline{x}_n, \bar{x}_n]\}. \quad (1a)$$

Computing this range for different algorithms  $f(x_1, \dots, x_n)$  is one of the main tasks of *interval computation*; see, e.g., [1, 2, 3, 4].

**Need for semi-infinite intervals.** The scale of each measuring instrument is bounded.

The lower value  $\ell$  on this scale does not mean that the actual value is close to  $\ell$ : it means that the actual value is less than or equal to  $\ell$ , i.e., that it belongs to the interval  $(-\infty, \ell]$ .

Similarly, when the instrument shows the upper value  $u$ , this means that the actual value is larger than or equal to  $u$ , i.e., that it belongs to the interval  $[u, \infty)$ .

**Interval arithmetic: reminder.** In the computer, every algorithms is represented as a sequence of basic arithmetic operations: addition, subtraction, multiplication, and division. To be more precise, division  $\frac{a}{b}$  is implemented as  $a \cdot \frac{1}{b}$ , so basic operations are, in effect, addition, subtraction, multiplication, and inversion  $\frac{1}{b}$ . Whatever we ask the computer to compute, be it  $\sin(x)$  or  $\ln(x)$ , the computer computes this value by using an appropriate sequence of these four hardware supported operations.

In view of this, not surprisingly, most algorithms of interval computation also build upon cases when the function (1) is one of these four arithmetic operations. One can easily check that the corresponding ranges can be described by the following expressions:

$$[\underline{a}, \bar{a}] + [\underline{b}, \bar{b}] = [\underline{a} + \underline{b}, \bar{a} + \bar{b}]; \quad (2)$$

$$[\underline{a}, \bar{a}] - [\underline{b}, \bar{b}] = [\underline{a} - \bar{b}, \bar{a} - \underline{b}]; \quad (3)$$

$$[\underline{a}, \bar{a}] \cdot [\underline{b}, \bar{b}] = [\min(\underline{a} \cdot \underline{b}, \underline{a} \cdot \bar{b}, \bar{a} \cdot \underline{b}, \bar{a} \cdot \bar{b}), \max(\underline{a} \cdot \underline{b}, \underline{a} \cdot \bar{b}, \bar{a} \cdot \underline{b}, \bar{a} \cdot \bar{b})]; \quad (4)$$

$$\frac{1}{[\underline{a}, \bar{a}]} = \left[ \frac{1}{\bar{a}}, \frac{1}{\underline{a}} \right] \text{ if } 0 \notin [\underline{a}, \bar{a}]. \quad (5)$$

These formulas are known as formulas of *interval arithmetic*.

**Case of semi-infinite intervals.** Formulas of interval arithmetic are applicable to semi-infinite intervals as well, if we use the usual calculus-based rules for dealing with infinities (and change closed bounds to open ones if this bound is plus or minus infinity). Namely, for all real numbers  $a$ :

$$\infty + a = \infty, \infty + \infty = \infty, (-\infty) + a = -\infty, (-\infty) + (-\infty) = -\infty, \quad (6)$$

$$\begin{aligned} \infty - a = \infty, \infty - (-\infty) = \infty, a - \infty = -\infty, (-\infty) - a = -\infty, \\ (-\infty) - \infty = -\infty. \end{aligned} \quad (7)$$

For multiplication, the only difference is in multiplication by 0:

- if  $a > 0$ , then

$$\infty \cdot a = \infty \text{ and } (-\infty) \cdot a = -\infty; \quad (8)$$

- if  $a < 0$ , then

$$\infty \cdot a = -\infty \text{ and } (-\infty) \cdot a = \infty; \quad (9)$$

- if  $a = 0$ , then

$$\infty \cdot a = (-\infty) \cdot a = 0. \quad (10)$$

For inverse, we have

$$\frac{1}{\infty} = \frac{1}{-\infty} = 0. \quad (11)$$

For  $1/0$ , we get either  $\infty$  or  $-\infty$ :

- for an interval  $[0, \bar{a}]$ , with  $0 < \bar{a}$ , we have

$$\frac{1}{[0, \bar{a}]} = \left[ \frac{1}{\bar{a}}, \infty \right); \tag{12}$$

- for an interval  $[\underline{a}, 0]$ , with  $\underline{a} < 0$ , we have

$$\frac{1}{[\underline{a}, 0]} = \left( -\infty, \frac{1}{\underline{a}} \right]. \tag{13}$$

The product formula can be described by the following table:

	$\underline{a} \leq \underline{a} \leq 0$	$\underline{a} \leq 0 \leq \bar{a}$	$0 \leq \underline{a} \leq \bar{a}$
$\underline{b} \leq \underline{b} \leq 0$	$[\bar{a} \cdot \underline{b}, \underline{a} \cdot \underline{b}]$	$[\bar{a} \cdot \underline{b}, \underline{a} \cdot \underline{b}]$	$[\bar{a} \cdot \underline{b}, \underline{a} \cdot \underline{b}]$
$\underline{b} \leq 0 \leq \bar{b}$	$[\underline{a} \cdot \underline{b}, \underline{a} \cdot \bar{b}]$	$[\min(\underline{a} \cdot \underline{b}, \bar{a} \cdot \underline{b}), \max(\underline{a} \cdot \bar{b}, \bar{a} \cdot \bar{b})]$	$[\bar{a} \cdot \underline{b}, \bar{a} \cdot \bar{b}]$
$0 \leq \underline{b} \leq \bar{b}$	$[\underline{a} \cdot \underline{b}, \bar{a} \cdot \bar{b}]$	$[\underline{a} \cdot \bar{b}, \bar{a} \cdot \bar{b}]$	$[\underline{a} \cdot \bar{b}, \bar{a} \cdot \bar{b}]$

**In interval arithmetic, inverse is not always defined.** In the usual interval arithmetic, inverse is defined only when  $0 \notin [\underline{a}, \bar{a}]$ . If we allow semi-infinite intervals, we can cover the cases when either  $\underline{a} = 0$  or  $\bar{a} = 0$ . But what if  $0 \in (\underline{a}, \bar{a})$ ? In this case, the range of  $1/a$  is a union of two disjoint intervals

$$\frac{1}{[\underline{a}, \bar{a}]} = \left( -\infty, \frac{1}{\underline{a}} \right] \cup \left[ \frac{1}{\bar{a}}, \infty \right). \tag{14}$$

**Formulation of the problem and what we do in this paper.** It is reasonable to want to extend interval arithmetic to operations with such unions. Such extensions are described in this paper.

*Comment.* Formula (14) only leads to unions  $(-\infty, a^-] \cup [a^+, \infty)$  when  $a^- < 0 < a^+$ . However, if we consider the sum of this set and a number  $b$ , then we get similar unions where  $a^-$  and  $a^+$  can be of the same sign. Thus, it is desirable to consider all possible unions of this type.

## 2 Main Results

**Main idea.** We want to consider ranges  $f(X_1)$  and  $f(X_1, X_2)$  in situations when one of the sets  $X_i$  (or both of them) is a union:  $X_i = X_i^- \cup X_i^+$ . In this case, by definition of the range, we have

$$f(X_1^- \cup X_1^+) = f(X_1^-) \cup f(X_1^+); \tag{15}$$

$$f(X_1^- \cup X_1^+, X_2) = f(X_1^-, X_2) \cup f(X_1^+, X_2); \tag{16}$$

$$f(X_1, X_2^- \cup X_2^+) = f(X_1, X_2^-) \cup f(X_1, X_2^+); \tag{17}$$

$$\begin{aligned} f(X_1^- \cup X_1^+, X_2^- \cup X_2^+) &= f(X_1^-, X_2^- \cup X_2^+) \cup f(X_1^+, X_2^- \cup X_2^+) = \\ &= f(X_1^-, X_2^-) \cup f(X_1^-, X_2^+) \cup f(X_1^+, X_2^-) \cup f(X_1^+, X_2^+). \end{aligned} \tag{18}$$

**Notations.** The fact that a value  $a$  belongs to the union  $(-\infty, a^-] \cup [a^+, \infty)$  means that it does not belong to the interval  $(a^-, a^+)$ . It is thus natural to call this union a *negative interval*. Correspondingly, usual intervals will be called *positive*.

To distinguish negative intervals from the usual ones, a natural idea is to swap the bounds, i.e., to denote this union by  $[a^+, a^-]$ .

In other words, when  $\underline{a} > \bar{a}$ , we define the set  $[\underline{a}, \bar{a}]$  as

$$[\underline{a}, \bar{a}] \stackrel{\text{def}}{=} (-\infty, \bar{a}] \cup [\underline{a}, \infty). \quad (19)$$

**How inverse of a normal interval looks in this notation.** In this notation, the formula (14) takes the form

$$\frac{1}{[\underline{a}, \bar{a}]} = \left[ \frac{1}{\bar{a}}, \frac{1}{\underline{a}} \right], \quad (20)$$

i.e., the same form as in the usual formula (5) – which can now be applied to all intervals  $[\underline{a}, \bar{a}]$ , whether they contain 0 or not.

**What we will do now.** We will now describe the formulas for arithmetic operations with negative intervals. Justifications of these formulas are given in the next section.

**The sum of a negative interval and a positive interval.** Let us first consider the case when:

- $[\underline{a}, \bar{a}]$  is a negative interval, i.e.,  $\underline{a} > \bar{a}$ , and
- $[\underline{b}, \bar{b}]$  is a positive interval, i.e.,  $\underline{b} \leq \bar{b}$ .

In this case:

- if  $\underline{a} + \underline{b} > \bar{a} + \bar{b}$ , then

$$[\underline{a}, \bar{a}] + [\underline{b}, \bar{b}] = [\underline{a} + \underline{b}, \bar{a} + \bar{b}]; \quad (21)$$

- otherwise, if  $\underline{a} + \underline{b} \leq \bar{a} + \bar{b}$ , then

$$[\underline{a}, \bar{a}] + [\underline{b}, \bar{b}] = \mathbb{R}, \quad (22)$$

where  $\mathbb{R}$  denotes the set of all real numbers.

In other words:

- we use the usual formula (2) for adding two intervals, if the result of applying this formula is a negative interval;
- if the result of applying the formula (2) is a positive interval, then the sum of negative and positive intervals is simply  $\mathbb{R}$ .

**The sum of two negative intervals.** The sum of two negative intervals is always the real line.

**The difference between a negative and a positive intervals.** When one of the intervals is negative and another one is positive, then:

- if  $\underline{a} - \bar{b} > \bar{a} - \underline{b}$ , then

$$[\underline{a}, \bar{a}] - [\underline{b}, \bar{b}] = [\underline{a} - \bar{b}, \bar{a} - \underline{b}]; \quad (23)$$

- otherwise, if  $\underline{a} - \bar{b} \leq \bar{a} - \underline{b}$ , then

$$[\underline{a}, \bar{a}] - [\underline{b}, \bar{b}] = \mathbb{R}. \tag{24}$$

In other words:

- we use the usual formula (3) for subtracting two intervals, if the result of applying this formula is a negative interval;
- if the result of applying the formula (3) is a positive interval, then the difference is simply  $\mathbb{R}$ .

**The difference between two negative intervals.** The difference between two negative intervals is the whole real line.

**Product of a negative interval and a real number.** Let

$$[\underline{a}, \bar{a}] = (-\infty, \bar{a}] \cup [\underline{a}, \infty)$$

be a negative interval, and let  $b$  be a real number. Then:

- when  $b > 0$ , then we get

$$[\underline{a}, \bar{a}] \cdot b = [b \cdot \underline{a}, b \cdot \bar{a}]; \tag{25}$$

- when  $b = 0$ , we get

$$[\underline{a}, \bar{a}] \cdot b = 0; \tag{26}$$

- when  $b < 0$ , then we get

$$[\underline{a}, \bar{a}] \cdot b = [b \cdot \bar{a}, b \cdot \underline{a}]. \tag{27}$$

**The product of a negative interval and a positive interval.** Let us consider the case when:

- $[\underline{a}, \bar{a}]$  is a negative interval, i.e.,  $\bar{a} < \underline{a}$ , and
- $[\underline{b}, \bar{b}]$  is a positive interval, i.e.,  $\underline{b} < \bar{b}$ .

Then, depending on the position of 0 with respect to these intervals, the product  $[\underline{a}, \bar{a}] \cdot [\underline{b}, \bar{b}]$  has the following form:

	$\bar{a} < \underline{a} \leq 0$	$\bar{a} < 0 < \underline{a}$	$0 \leq \bar{a} < \underline{a}$
$\underline{b} < \bar{b} < 0$	$[\bar{a} \cdot \bar{b}, \underline{a} \cdot \underline{b}]^*$	$[\bar{a} \cdot \bar{b}, \underline{a} \cdot \bar{b}]$	$[\bar{a} \cdot \underline{b}, \underline{a} \cdot \bar{b}]^*$
$\underline{b} < \bar{b} = 0$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$
$\underline{b} < 0 < \bar{b}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$
$0 = \underline{b} < \bar{b}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$
$0 < \underline{b} < \bar{b}$	$[\underline{a} \cdot \bar{b}, \bar{a} \cdot \underline{b}]^*$	$[\underline{a} \cdot \underline{b}, \bar{a} \cdot \underline{b}]$	$[\underline{a} \cdot \underline{b}, \bar{a} \cdot \bar{b}]^*$

Here,  $[a, b]^*$  means that if  $[a, b]$  is not a negative interval, i.e., if  $a \leq b$ , then the result is the real line  $\mathbb{R}$ .

*Comment.* Interestingly, in the four corner cells, we get the same expression as in the table for the product of two positive intervals, but the expressions in the middle column are different.

**The product of two negative intervals.** If  $\bar{a} < 0 < \underline{a}$  and  $\bar{b} < 0 < \underline{b}$ , then

$$[\underline{a}, \bar{a}] \cdot [\underline{b}, \bar{b}] = [\min(\underline{a} \cdot \underline{b}, \bar{a} \cdot \bar{b}), \max(\underline{a} \cdot \bar{b}, \bar{a} \cdot \underline{b})].$$

In all other cases, the product is equal to the whole real line.

*Comment.* Interestingly, when the product of two negative intervals is not the whole real line, it is described by the same formula as the product of two positive intervals.

**The inverse of a negative interval: discussion.** When a quantity  $a$  takes all possible values from a negative interval

$$[1, -1] = (-\infty, -1] \cup [1, \infty),$$

then the set of possible value of the inverse  $\frac{1}{a}$  is a union

$$[-1, 0) \cup (0, 1].$$

It is *almost* the interval  $[-1, 1]$ ; the only difference is that the value 0 does not belong to the range, since 0 cannot be obtained as  $\frac{1}{a}$  for a real number  $a$ . In this case, instead of the actual range, it makes sense to consider the *interval hull*  $[-1, 1]$  of the range, i.e., the intersection of all intervals containing the range.

Similarly, when a quantity  $a$  takes all possible values from a negative interval

$$[2, 1] = (-\infty, 1] \cup [2, \infty),$$

then the set of possible value of the inverse  $\frac{1}{a}$  is a union

$$(-\infty, 0) \cup (0, 0.5] \cup [1, \infty).$$

It is *almost* the negative interval

$$[1, 0.5] = (-\infty, 0.5] \cup [1, \infty);$$

the only difference is that the value 0 does not belong to the range, since 0 cannot be obtained as  $\frac{1}{a}$  for a real number  $a$ . In this case, instead of the actual range, it makes sense to consider the *negative-interval hull*  $[1, 0.5]$  of the range, i.e., the intersection of all negative interval containing the range.

**The inverse of a negative interval: results.** We will denote such equality modulo number 0 by  $\stackrel{0}{=}$ . For this equality, we get a formula very similar to the usual formula (5):

$$\frac{1}{[\underline{a}, \bar{a}]} \stackrel{0}{=} \left[ \frac{1}{\bar{a}}, \frac{1}{\underline{a}} \right]. \quad (28)$$

In contrast to the original formula (5), this formula is applicable always, whether 0 belongs to the original negative interval or not.

## 3 Proofs

### 3.1 The sum of a negative interval and a positive interval

Here,

$$\begin{aligned} [\underline{a}, \bar{a}] + [\underline{b}, \bar{b}] &= ((-\infty, \bar{a}] \cup [\underline{a}, \infty)) + [\underline{b}, \bar{b}] = \\ &= ((-\infty, \bar{a}] + [\underline{b}, \bar{b}]) \cup ([\underline{a}, \infty) + [\underline{b}, \bar{b}]). \end{aligned} \quad (29)$$

By the formula (2), taking into account operations with infinities, the first sum in the formula (29) takes the form

$$(-\infty, \bar{a}] + [\underline{b}, \bar{b}] = (-\infty, \bar{a} + \bar{b}]. \quad (30)$$

The second sum in the formula (29) takes the form

$$[\underline{a}, \infty) + [\underline{b}, \bar{b}] = [\underline{a} + \underline{b}, \infty). \quad (31)$$

Thus, their union takes the form

$$[\underline{a}, \bar{a}] + [\underline{b}, \bar{b}] = (-\infty, \bar{a} + \bar{b}] \cup [\underline{a} + \underline{b}, \infty). \quad (32)$$

If  $\bar{a} + \bar{b} < \underline{a} + \underline{b}$ , then the united semi-intervals are disjoint, i.e., we have the desired negative interval. Otherwise, if  $\bar{a} + \bar{b} \geq \underline{a} + \underline{b}$ , the union is the whole real line.

### 3.2 The sum of two negative intervals

The sum of two negative intervals has the following form:

$$[\underline{a}, \bar{a}] + [\underline{b}, \bar{b}] = ((-\infty, \bar{a}] \cup [\underline{a}, \infty)) + ((-\infty, \bar{a}] \cup [\underline{a}, \infty)) =$$

$$((-\infty, \bar{a}] + (-\infty, \bar{b}]) \cup ((-\infty, \bar{a}] + [\underline{b}, \infty)) \cup ([\underline{a}, \infty) + [\underline{b}, \infty)) \cup ([\underline{a}, \infty) + [\underline{b}, \infty)).$$

Here,

$$(-\infty, \bar{a}] + [\underline{b}, \infty) = (-\infty + \underline{b}, \bar{a} + \infty) = (-\infty, \infty) = \mathbb{R}.$$

Thus, the union of this sum with other sets is also the whole real line  $\mathbb{R}$ .

### 3.3 The difference between a negative and a positive intervals

This case can be reduced to the case of the sum if we take into account:

- that  $a - b = a + c$ , where we denoted  $c \stackrel{\text{def}}{=} -b$ , and
- that if the set of possible values of  $b$  is the interval  $[\underline{b}, \bar{b}]$ , then the set of possible values of  $c = -b$  is  $[\underline{c}, \bar{c}] = [-\bar{b}, -\underline{b}]$ .

By applying the sum formula to the intervals for  $a$  and  $c$ , we get the desired result.

### 3.4 The difference between a positive interval and a negative interval

This case can also be reduced to the sum if we take into account:

- that  $a - b = a + c$ , where we denoted  $c \stackrel{\text{def}}{=} -b$ , and
- that if the set of possible values of  $b$  is the negative interval

$$[\underline{b}, \bar{b}] = (-\infty, \bar{b}] \cup [\underline{b}, \infty),$$

then the set of possible values of  $c = -b$  is/

$$(-\infty, -\underline{b}] \cup [-\bar{b}, \infty) = [-\bar{b}, -\underline{b}].$$

By applying the sum formula to the intervals for  $a$  and  $c$ , we get the desired result.



### 3.5 The difference between two negative intervals

Here,

$$[\underline{a}, \bar{a}] - [\underline{b}, \bar{b}] = ((-\infty, \bar{a}] \cup [\underline{a}, \infty)) - ((-\infty, \bar{a}] \cup [\underline{a}, \infty)) = \\ ((-\infty, \bar{a}] - (-\infty, \bar{b}]) \cup ((-\infty, \bar{a}] - [\underline{b}, \infty)) \cup ([\underline{a}, \infty) - [\underline{b}, \infty)) \cup ([\underline{a}, \infty) - [\underline{b}, \infty)).$$

In this case,

$$(-\infty, \bar{a}] - (-\infty, \bar{b}] = (-\infty - \bar{b}, \underline{b} - (-\infty)) = (-\infty, \infty) = \mathbb{R}.$$

Thus, the union of this sum with other sets is also the whole real line  $\mathbb{R}$ .

### 3.6 The product of a negative interval and a real number

**Case when  $b > 0$ .** When  $b > 0$ , then

$$(-\infty, \bar{a}] \cdot b = (-\infty, \bar{a}] \cdot [b, b] = [\min(-\infty, \bar{a} \cdot b), \max(-\infty, \bar{a} \cdot b)] = (-\infty, \bar{a} \cdot b]$$

and

$$[\underline{a}, \infty) \cdot b = [\underline{a}, \infty) \cdot [b, b] = [\min(\underline{a} \cdot b, \infty), \max(\underline{a} \cdot b, \infty)] = [\underline{a} \cdot b, \infty).$$

The union of these two semi-infinite intervals leads to the desired result.

**Case when  $b = 0$ .** The product of any number and  $b = 0$  is 0, so we have  $[\underline{a}, \bar{a}] \cdot b = 0$ .

**Case when  $b < 0$ .** When  $b < 0$ , then

$$(-\infty, \bar{a}] \cdot b = (-\infty, \bar{a}] \cdot [b, b] = [\min(\infty, \bar{a} \cdot b), \max(\infty, \bar{a} \cdot b)] = [\bar{a} \cdot b, \infty)$$

and

$$[\underline{a}, \infty) \cdot b = [\underline{a}, \infty) \cdot [b, b] = [\min(\underline{a} \cdot b, -\infty), \max(\underline{a} \cdot b, -\infty)] = (-\infty, \underline{a} \cdot b].$$

The union of these two semi-infinite intervals leads to the desired result.

### 3.7 The product of a negative interval and a positive interval

In general,

$$[\underline{a}, \bar{a}] \cdot [\underline{b}, \bar{b}] = ((-\infty, \bar{a}] \cup [\underline{a}, \infty)) \cdot [\underline{b}, \bar{b}] = ((-\infty, \bar{a}] \cdot [\underline{b}, \bar{b}]) \cup ([\underline{a}, \infty) \cdot [\underline{b}, \bar{b}]).$$

In line with the above table, we will consider  $3 \cdot 5 = 15$  cases.

**Case 1.1:**  $\bar{a} < \underline{a} \leq 0$  and  $\underline{b} < \bar{b} < 0$ . In this case, since  $\bar{a} < 0$ , we have  $\bar{a} \cdot \bar{b} < \bar{a} \cdot \underline{b}$ , so

$$(-\infty, \bar{a}] \cdot [\underline{b}, \bar{b}] = [\min(\infty, \bar{a} \cdot \underline{b}, \bar{a} \cdot \bar{b}), \max(\infty, \bar{a} \cdot \underline{b}, \bar{a} \cdot \bar{b})] = [\bar{a} \cdot \bar{b}, \infty).$$

Similarly, since  $\underline{a} \leq 0$ , we have  $\underline{a} \cdot \bar{b} \leq \underline{a} \cdot \underline{b}$ , so

$$[\underline{a}, \infty) \cdot [\underline{b}, \bar{b}] = [\min(-\infty, \underline{a} \cdot \underline{b}, \underline{a} \cdot \bar{b}), \max(-\infty, \underline{a} \cdot \underline{b}, \underline{a} \cdot \bar{b})] = (-\infty, \underline{a} \cdot \underline{b}].$$

Thus, the union of these products is equal to

$$(-\infty, \underline{a} \cdot \underline{b}] \cup [\bar{a} \cdot \bar{b}, \infty).$$

- If  $\underline{a} \cdot \underline{b} < \bar{a} \cdot \bar{b}$ , then this union is a negative interval  $[\bar{a} \cdot \bar{b}, \underline{a} \cdot \underline{b}]$ .
- Otherwise, if  $\bar{a} \cdot \bar{b} \leq \underline{a} \cdot \underline{b}$ , then this union coincides with the whole real line  $\mathbb{R}$ .

**Case 1.2:**  $\bar{a} < \underline{a} \leq 0$  and  $\underline{b} < \bar{b} = 0$ . In this case, since  $\bar{a} < 0$ , we have  $0 < \bar{a} \cdot \underline{b}$ , so

$$(-\infty, \bar{a}] \cdot [\underline{b}, 0] = [\min(\infty, \bar{a} \cdot \underline{b}, 0), \max(\infty, \bar{a} \cdot \underline{b}, 0)] = [0, \infty).$$

Similarly, since  $\underline{a} \leq 0$ , we have  $0 \leq \underline{a} \cdot \underline{b}$ , so

$$[\underline{a}, \infty) \cdot [\underline{b}, 0] = [\min(-\infty, \underline{a} \cdot \underline{b}, 0), \max(-\infty, \underline{a} \cdot \underline{b}, 0)] = (-\infty, \underline{a} \cdot \underline{b}].$$

Thus, the union of these products is equal to

$$(-\infty, \underline{a} \cdot \underline{b}] \cup [0, \infty).$$

Here,  $\underline{a} < 0$ ,  $\underline{b} < 0$ , hence  $\underline{a} \cdot \underline{b} > 0$  and thus, the above union is equal to the real line.

**Case 1.3:**  $\bar{a} < \underline{a} \leq 0$  and  $\underline{b} < 0 < \bar{b}$ . In this case, already the first product is the real line:

$$(-\infty, \bar{a}] \cdot [\underline{b}, \bar{b}] = [\min(\infty, \bar{a} \cdot \underline{b}, \bar{a} \cdot \bar{b}, \infty), \max(\infty, \bar{a} \cdot \underline{b}, \bar{a} \cdot \bar{b}, \infty)] = (-\infty, \infty),$$

so the union of the two products is also the real line.

**Case 1.4:**  $\bar{a} < \underline{a} \leq 0$  and  $\underline{b} < \bar{b} = 0$ . In this case, for  $c = -b$ , the set of possible values if  $[\underline{c}, \bar{c}] = [-\bar{b}, -\underline{b}]$ , so  $\underline{c} = 0 < \bar{c}$ . In general, we have  $a \cdot b = -(a \cdot (-b))$ , so for any sets  $A$  and  $B$  of possible values of  $a$  and  $b$ , we have  $A \cdot B = -(A \cdot (-B)) = -(A \cdot C)$ . Due to Case 1.2, here  $A \cdot C = \mathbb{R}$ , so we have

$$A \cdot B = -\mathbb{R} = \{-x : x \in \mathbb{R}\} = \mathbb{R}.$$

**Case 1.5:**  $\bar{a} < \underline{a} \leq 0$  and  $\underline{b} < \bar{b} < 0$ . In this case also, for  $c = -b$ , we have  $A \cdot B = -(A \cdot (-B)) = -(A \cdot C)$ . Due to Case 1.1, here

$$A \cdot C = [\bar{a} \cdot \bar{c}, \underline{a} \cdot \underline{c}] = (-\infty, \underline{a} \cdot \underline{c}] \cup [\bar{a} \cdot \bar{c}, \infty) = (-\infty, -\underline{a} \cdot \bar{b}] \cup [-\bar{a} \cdot \underline{b}, \infty).$$

Thus, for  $A \cdot B = -(A \cdot C)$ , we have

$$-(-\infty, -\underline{a} \cdot \bar{b}] = [-1, -1] \cdot (-\infty, -\underline{a} \cdot \bar{b}] = [\underline{a} \cdot \bar{b}, \infty)$$

and

$$-[-\bar{a} \cdot \underline{b}, \infty) = [-1, -1] \cdot [-\bar{a} \cdot \underline{b}, \infty) = [\bar{a} \cdot \underline{b}, \infty).$$

Thus,

$$A \cdot B = (-\infty, \bar{a} \cdot \underline{b}] \cup [\underline{a} \cdot \bar{b}, \infty) = [\underline{a} \cdot \bar{b}, \bar{a} \cdot \underline{b}]$$

if this is a negative interval – otherwise, it is the whole real line.

**Case 2.1:**  $\bar{a} < 0 < \underline{a}$  and  $\underline{b} < \bar{b} < 0$ . In this case, since  $\bar{a} < 0$ , we have  $\bar{a} \cdot \bar{b} < \bar{a} \cdot \underline{b}$ , so

$$(-\infty, \bar{a}] \cdot [\underline{b}, \bar{b}] = [\min(\infty, \bar{a} \cdot \underline{b}, \bar{a} \cdot \bar{b}), \max(\infty, \bar{a} \cdot \underline{b}, \bar{a} \cdot \bar{b})] = [\bar{a} \cdot \bar{b}, \infty).$$

Since  $\underline{a} > 0$ , we have  $\underline{a} \cdot \underline{b} < \underline{a} \cdot \bar{b}$ , so

$$[\underline{a}, \infty) \cdot [\underline{b}, \bar{b}] = [\min(-\infty, \underline{a} \cdot \underline{b}, \underline{a} \cdot \bar{b}), \min(-\infty, \underline{a} \cdot \underline{b}, \underline{a} \cdot \bar{b})] = (-\infty, \underline{a} \cdot \bar{b}].$$

Thus, the union of these two products is equal to:

$$(-\infty, \underline{a} \cdot \bar{b}] \cup [\bar{a} \cdot \bar{b}, \infty) = [\bar{a} \cdot \bar{b}, \underline{a} \cdot \bar{b}].$$

**Case 2.2:**  $\bar{a} < 0 < \underline{a}$  and  $\underline{b} < \bar{b} = 0$ . In this case, since  $\bar{a} < 0$ , we have  $0 < \bar{a} \cdot \underline{b}$ , so

$$(-\infty, \bar{a}] \cdot [\underline{b}, \bar{b}] = [\min(\infty, \bar{a} \cdot \underline{b}, 0), \max(\infty, \bar{a} \cdot \underline{b}, 0)] = [0, \infty).$$

Since  $\underline{a} > 0$ , we have  $\underline{a} \cdot \underline{b} < 0$ , so

$$[\underline{a}, \infty) \cdot [\underline{b}, \bar{b}] = [\min(-\infty, \underline{a} \cdot \underline{b}, 0), \min(-\infty, \underline{a} \cdot \underline{b}, 0)] = (-\infty, 0].$$

Thus, the union of these two products  $(-\infty, 0]$  and  $[0, \infty)$  is equal to the whole real line.

**Case 2.3:**  $\bar{a} < 0 < \underline{a}$  and  $\underline{b} < 0 < \bar{b}$ . In this case, already the first product is the real line:

$$(-\infty, \bar{a}] \cdot [\underline{b}, \bar{b}] = [\min(\infty, -\infty, \bar{a} \cdot \underline{b}, \bar{a} \cdot \bar{b}), \max(\infty, -\infty, \bar{a} \cdot \underline{b}, \bar{a} \cdot \bar{b})] = (-\infty, \infty),$$

so the union of the two products is also the real line.

**Case 2.4:**  $\bar{a} < 0 < \underline{a}$  and  $0 = \underline{b} < \bar{b}$ . By using the formula  $a \cdot b = -(a \cdot (-b))$ , we reduce this case to Case 2.2. So, similarly to Case 1.4, we get the real line.

**Case 2.5:**  $\bar{a} < 0 < \underline{a}$  and  $0 < \underline{b} < \bar{b}$ . By using the formula  $a \cdot b = -(a \cdot c)$ , for  $c = -b$ , we reduce this case we reduce this case to Case 2.1. So,

$$\begin{aligned} A \cdot B &= -((-\infty, \underline{a} \cdot \bar{c}] \cup [\bar{a} \cdot \bar{c}, \infty)) = -((-\infty, -\underline{a} \cdot \underline{b}] \cup [-\bar{a} \cdot \bar{b}, \infty)) = \\ &(-\infty, \bar{a} \cdot \bar{b}] \cup [\underline{a} \cdot \underline{b}, \infty) = [\underline{a} \cdot \underline{b}, \bar{a} \cdot \bar{b}]. \end{aligned}$$

**Case 3.1:**  $0 \leq \bar{a} < \underline{a}$  and  $\underline{b} < \bar{b} < 0$ . In this case,  $a \cdot b = -(c \cdot b)$ , where we denoted  $c \stackrel{\text{def}}{=} -a$ , so  $A \cdot B = -((-A) \cdot B)$ . When

$$a \in A = [\underline{a}, \bar{a}] = (-\infty, \bar{a}] \cup [\underline{a}, \infty),$$

then

$$c \in C = -A = (-\infty, -\underline{a}] \cup [-\bar{a}, \infty) = [-\bar{a}, -\underline{a}].$$

Here, due to the formulas from Case 1.1:

$$C \cdot B = (-\infty, \underline{c} \cdot \underline{b}] \cup [\bar{a} \cdot \bar{c}, \infty) = (-\infty, -\bar{a} \cdot \underline{b}] \cup [-\underline{a} \cdot \bar{b}, \infty),$$

so

$$A \cdot B = -(C \cdot B) = (-\infty, \underline{a} \cdot \bar{b}] \cup [\bar{a} \cdot \underline{b}, \infty).$$

Similarly, to Case 1.1, this is either a negative interval  $[\bar{a} \cdot \underline{b}, \underline{a} \cdot \bar{b}]$ , or – if the first number in this expression is not larger than the second one – the whole real line.

**Case 3.2:**  $0 \leq \bar{a} < \underline{a}$  and  $\underline{b} < \bar{b} = 0$ . This case can be obtained by the same reduction  $A \cdot B = -((-A) \cdot B)$  from Case 1.2, so we have  $A \cdot B = -\mathbb{R} = \mathbb{R}$ .

**Case 3.3:**  $0 \leq \bar{a} < \underline{a}$  and  $\underline{b} < 0 < \bar{b}$ . This case can be obtained by the same reduction from Case 1.3, so we have  $A \cdot B = -((-A) \cdot B) = -\mathbb{R} = \mathbb{R}$ .

**Case 3.4:**  $0 \leq \bar{a} < \underline{a}$  and  $\underline{b} = 0 < \bar{b}$ . This case can be obtained by the same reduction from Case 1.4, so we have  $A \cdot B = -((-A) \cdot B) = -\mathbb{R} = \mathbb{R}$ .

**Case 3.5:**  $0 \leq \bar{a} < \underline{a}$  and  $0 < \underline{b} < \bar{b}$ . This case can be obtained by the same reduction  $A \cdot B = -((-A) \cdot B)$  from Case 1.5. Due to this case, for

$$C = -A = -((-\infty, \bar{a}] \cup [\underline{a}, \infty)) = (-\infty, -\underline{a}] \cup [-\bar{a}, \infty) = [-\bar{a}, -\underline{a}],$$

we have

$$(-A) \cdot B = C \cdot B = (-\infty, \bar{c} \cdot \bar{b}] \cup [\underline{c} \cdot \bar{b}, \infty) = (-\infty, -\underline{a} \cdot \bar{b}] \cup [-\bar{a} \cdot \bar{b}, \infty),$$

so

$$A \cdot B = -(A \cdot C) = (-\infty, \bar{a} \cdot \bar{b}] \cup [\underline{a} \cdot \bar{b}, \infty)$$

which is either a negative interval  $[\underline{a} \cdot \bar{b}, \bar{a} \cdot \bar{b}]$  or – if this expression is not a negative interval – the whole real line.

### 3.8 The product of two negative intervals

In general:

$$\begin{aligned} [\underline{a}, \bar{a}] \cdot [\underline{b}, \bar{b}] &= ((-\infty, \bar{a}] \cup [\underline{a}, \infty)) \cdot ((-\infty, \bar{a}] \cup [\underline{a}, \infty)) = \\ &((-\infty, \bar{a}] \cdot (-\infty, \bar{b}]) \cup ((-\infty, \bar{a}] \cdot [b, \infty)) \cup ([\underline{a}, \infty) \cdot (-\infty, \bar{b}]) \cup ([\underline{a}, \infty) \cdot [b, \infty)). \end{aligned}$$

If  $\bar{a} < 0 < \underline{a}$  and  $\bar{b} < 0 < \underline{b}$ , then, since  $\bar{a} < 0$  and  $\bar{b} < 0$ , we have  $\infty \cdot \bar{a} = \infty \cdot \bar{b} = \infty$ . So, the first of the four united sets is:

$$[\min(\infty, \infty, \infty, \bar{a} \cdot \bar{b}), \max(\infty, \infty, \infty, \bar{a} \cdot \bar{b})] = [\bar{a} \cdot \bar{b}, \infty).$$

Similarly, the second product has the form  $(-\infty, \bar{a} \cdot \bar{b}]$ , the third has the form  $(-\infty, \bar{a} \cdot \bar{b}]$ , and the fourth the form  $[\underline{a} \cdot \bar{b}, \infty)$ . Thus, the union of these four intervals is equal to

$$\begin{aligned} (-\infty, \max(\bar{a} \cdot \bar{b}, \bar{a} \cdot \bar{b})] \cup [\min(\underline{a} \cdot \bar{b}, \bar{a} \cdot \bar{b}), \infty) = \\ [\min(\underline{a} \cdot \bar{b}, \bar{a} \cdot \bar{b}), \max(\bar{a} \cdot \bar{b}, \bar{a} \cdot \bar{b})]. \end{aligned}$$

Let us show that when at least of one of the negative intervals, 0 is not strictly inside the bounds, then the product is the whole real line. Without losing generality, we can assume that 0 is not strictly inside the bounds for the  $a$ -interval.

In this case, either both bounds of the  $a$ -interval are non-negative, or both bounds are non-positive. The case when both bounds are non-positive can be reduced to the case when they are both non-negative by taking into account that  $A \cdot (-B) = -(A \cdot B)$ . So, it is sufficient to consider the case when both bounds are non-negative, i.e., that  $0 \leq \bar{a} < \underline{a}$ .

If  $\bar{a} = 0$ , then the above formula applies, and it leads to  $(-\infty, 0] \cup [0, \infty) = \mathbb{R}$ .

If  $\bar{a} > 0$ , then  $(-\infty) \cdot (-\infty) = \infty$  and  $\bar{a} \cdot (-\infty) = -\infty$ , thus the first product is the whole real line

$$(-\infty, \bar{a}] \cdot (-\infty, \bar{b}] = [\min(-\infty, \infty, \dots), \max(-\infty, \infty, \dots)] = (-\infty, \infty).$$

Thus, its union with other sets is also the real line.

### 3.9 The inverse of a negative interval

**Possible cases.** We want to find the inverse of a negative interval

$$[\underline{a}, \bar{a}] = (-\infty, \bar{a}] \cup [\underline{a}, \infty). \quad (33)$$

For usual intervals, the inverse changes depending on whether 0 is inside the interval or not. So, let us consider three possible cases:  $\bar{a} < 0 < \underline{a}$ ,  $0 \leq \bar{a} < \underline{a}$ , and  $\bar{a} < \underline{a} \leq 0$ .

**Case when  $\bar{a} < 0 < \underline{a}$ .** In this case, we have

$$\frac{1}{(-\infty, \bar{a}]} = \left[ \frac{1}{\bar{a}}, 0 \right) \quad (34)$$

and

$$\frac{1}{[\underline{a}, \infty)} = \left( 0, \frac{1}{\underline{a}} \right]. \quad (35)$$

The union of these two intervals is almost an interval  $\left[ \frac{1}{\bar{a}}, \frac{1}{\underline{a}} \right]$ , except that it does not contain 0. We can compute the interval hull, i.e., the intersection of all intervals containing this union, then we get:

$$\frac{1}{[\underline{a}, \bar{a}]} \stackrel{0}{=} \left[ \frac{1}{\bar{a}}, \frac{1}{\underline{a}} \right]. \quad (36)$$

**Case when  $0 \leq \bar{a} < \underline{a}$ .** Since the interval inverse formula is only available when 0 is not in the interior of the interval, we have to represent the negative interval (33) as a union of three intervals:

$$[\underline{a}, \bar{a}] = (-\infty, 0] \cup [0, \bar{a}] \cup [\underline{a}, \infty), \quad (37)$$

so that

$$\frac{1}{[\underline{a}, \bar{a}]} = \frac{1}{(-\infty, 0]} \cup \frac{1}{[0, \bar{a}]} \cup \frac{1}{[\underline{a}, \infty)}. \quad (38)$$

Here,

$$\frac{1}{(-\infty, 0]} = \left( \frac{1}{0}, \frac{1}{-\infty} \right] = (-\infty, 0), \quad (39)$$

$$\frac{1}{[0, \bar{a}]} = \left[ \frac{1}{\bar{a}}, \infty \right), \quad (40)$$

and

$$\frac{1}{[\underline{a}, \infty)} = \left( 0, \frac{1}{\underline{a}} \right]. \quad (41)$$

Thus, the union of these three intervals is equal to

$$(-\infty, 0) \cup \left( 0, \frac{1}{\underline{a}} \right] \cup \left[ \frac{1}{\bar{a}}, \infty \right).$$

This result is almost equal to the negative interval

$$\left( -\infty, \frac{1}{\underline{a}} \right] \cup \left[ \frac{1}{\bar{a}}, \infty \right) = \left[ \frac{1}{\bar{a}}, \frac{1}{\underline{a}} \right].$$

So, in this case, we also get the formula (28).

**Case when  $\underline{a} < \bar{a} \leq 0$ .** Since the interval inverse formula is only available when 0 is not in the interior of the interval, we have to represent the negative interval (19) as a union of three intervals:

$$[\underline{a}, \bar{a}] = (-\infty, \bar{a}] \cup [\underline{a}, 0] \cup [0, \infty), \tag{42}$$

so that

$$\frac{1}{[\underline{a}, \bar{a}]} = \frac{1}{(-\infty, \bar{a}]} \cup \frac{1}{[\underline{a}, 0]} \cup \frac{1}{[0, \infty)}. \tag{43}$$

Here,

$$\frac{1}{(-\infty, \bar{a}]} = \left[ \frac{1}{\bar{a}}, 0 \right), \tag{44}$$

$$\frac{1}{[\underline{a}, 0]} = \left( -\infty, \frac{1}{\underline{a}} \right], \tag{45}$$

and

$$\frac{1}{[0, \infty)} = \left[ \frac{1}{\infty}, \frac{1}{0} \right) = (0, \infty). \tag{46}$$

Thus, the union of these three intervals is equal to

$$\left( -\infty, \frac{1}{\underline{a}} \right] \cup \left[ \frac{1}{\bar{a}}, 0 \right) \cup (0, \infty) = \left( -\infty, \frac{1}{\underline{a}} \right] \cup \left[ \frac{1}{\bar{a}}, \infty \right). \tag{47}$$

So, in this case, considering the negative-interval hull, we get

$$\frac{1}{[\underline{a}, \bar{a}]} \stackrel{0}{=} \left[ \frac{1}{\bar{a}}, \frac{1}{\underline{a}} \right],$$

i.e., also the formula (28).

**Conclusion.** In all three cases, we have the same formula (28).

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