

Which Classes of Bi-Intervals Are Closed Under Addition? Under Linear Combination? Under Other Operations?*

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Abstract

In many practical situations, uncertainty with which we know each quantity is described by an interval. Techniques for processing such interval uncertainty use the fact that the sum, difference, and product of two intervals is always an interval. In some cases, the set of all possible values of a quantity is described by a *bi-interval* – i.e., by a union of two disjoint intervals. It is known that already the sum of two bi-intervals is not always a bi-interval. In this paper, we describe all the classes of bi-intervals which are closed under addition (i.e., for which the sum of bi-intervals is a bi-interval), closed under linear combination, and closed under other operations.

Keywords: interval uncertainty, union of two intervals, closeness under addition, closeness under linear combination, closeness under other operations

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1 Formulation of the Problem

Interval uncertainty: a brief reminder. In many real-life situations, our uncertainty about a quantity is described by an interval; see, e.g., [20].

For example, usually, the information about a physical quantity x comes from measurements. As a result of the measurement, we get a value \tilde{x} which is, in general, different from the actual (unknown) value x : $\Delta x \stackrel{\text{def}}{=} \tilde{x} - x \neq 0$. Often, the only information that we have about the measurement accuracy is the upper bound Δ on the absolute value $|\Delta x|$ of the measurement error Δx . In this case, after the measurement, the only information that we gain about the actual value x is that x belongs to the interval $[\tilde{x} - \Delta, \tilde{x} + \Delta]$; see, e.g., [20].

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Comments.

- In some cases, we only get the lower bound. In such cases, we get an interval $[v, \infty)$. It is also possible to have intervals $(-\infty, v]$, and, of course, we can have the whole real line $(-\infty, \infty)$ corresponding to the case when we do not have any information about x .
- Throughout this paper, intervals will be denoted by boldface letters.

Interval computations. In many practical situations, we want to understand the state of the world and we want to predict the future state of the world. The state of the world is characterized by the values of different quantities.

Some of these quantities we can directly measure, but many others are difficult to measure directly. So, we need to be able to estimate the value y of a difficult-to-directly-measure quantity. For prediction, we need to estimate the future values of different quantities y – and, of course, it is not possible to measure this future value now.

The usual way to perform these estimations is to use a known relation $y = f(x_1, \dots, x_n)$ between the desired quantity y and one or several directly measurable quantities x_1, \dots, x_n . So, to compute an estimate \tilde{y} for the desired quantity y , we measure the corresponding quantities x_i and then apply the algorithm f to the measurement results $\tilde{x}_1, \dots, \tilde{x}_n$: $\tilde{y} = f(\tilde{x}_1, \dots, \tilde{x}_n)$.

As we have mentioned, the actual (unknown) value of each quantity x_i is, in general, different from the measurement result \tilde{x}_i . As a result, even when the dependence $y = f(x_1, \dots, x_n)$ is known exactly, the actual value y is, in general, different from the estimate \tilde{y} . How different can it be? To answer this question, we need to find the range

$$\mathbf{y} \stackrel{\text{def}}{=} f(\mathbf{x}_1, \dots, \mathbf{x}_n) = \{f(x_1, \dots, x_n) : x_1 \in \mathbf{x}_1, \dots, x_n \in \mathbf{x}_n\} \quad (1)$$

of all possible values of $y = f(x_1, \dots, x_n)$ when all we know about each quantity x_i is that it belongs to the corresponding interval \mathbf{x}_i . The problem of computing this range is known as the problem of *interval computation*; see, e.g., [10, 15, 18]

Most algorithms for solving this problem use the fact that inside the computer, the only hardware supported numerical operations are elementary arithmetic operations: addition, subtraction, multiplication, and division. To be more precise, division a/b is implemented as the product $a \cdot (1/b)$, so what is hardware supported is not division itself, but the operation of taking the inverse $1/b$. Every computation inside the computer consists of a sequence of such arithmetic operations. For example, when a computer computes the value of $\exp(x)$ or $\sin(x)$, what it actually computes is the value of the corresponding polynomial – the sum of the first few terms in the corresponding Taylor expansion.

Since elementary arithmetic operations form the basis of all computations, it is natural to first solve the interval computation problem for the cases when the corresponding function $f(x_1, x_2)$ is one these arithmetic operations: addition, subtraction, multiplication, and taking the inverse. It is known that for these functions, the range \mathbf{y} is also an interval (expect, of course, for the case of the inverse $1/b$ when the interval of possible values of b contain 0. For example, if \mathbf{x}_1 and \mathbf{x}_2 are intervals, then their sum

$$\mathbf{x}_1 + \mathbf{x}_2 \stackrel{\text{def}}{=} \{x_1 + x_2 : x_1 \in \mathbf{x}_1 \text{ and } x_2 \in \mathbf{x}_2\} \quad (2)$$

is also an interval, and for every two real numbers c_0 and c_1 , the set

$$c_0 + c_1 \cdot \mathbf{x}_1 \stackrel{\text{def}}{=} \{c_0 + c_1 \cdot x_1 : x_1 \in \mathbf{x}_1\} \quad (3)$$

is also an interval. The corresponding range can be computed by using simple formulas; these formulas are known as formulas of *interval arithmetic*.

To estimate the range (1) for a generic algorithm $f(x_1, \dots, x_n)$ – that consists of a sequence of elementary arithmetic steps – as a very crude approximation, we can simply replace each operation with numbers with the corresponding operation with intervals. In some simple cases, this naive approach leads to reasonable results, but in general, to get meaningful results, we need to apply some sophisticated methods. In effect, these methods involve replacing the original algorithms with an equivalent one, and then kind-of applying interval arithmetic to the resulting algorithm. So, interval arithmetic still forms the basis for most current interval computations algorithms.

Linearization. In general, the problem of computing the range $f(\mathbf{x}_1, \dots, \mathbf{x}_n)$ is NP-hard already for quadratic functions $f(x_1, \dots, x_n)$ – the simplest non-linear functions; see, e.g., [12]. Crudely speaking, this means that unless $P=NP$ (which most computer scientists believe to be impossible), no feasible algorithm is possible that would always compute the exact range (1). In situations when we cannot compute the exact range and cannot compute a good *enclosure* (*outer approximation*) $\mathbf{Y} \supseteq \mathbf{y}$ for this range, a natural idea is to compute some approximate range.

A typical way to compute such an approximate range is to use the fact that measurements are usually reasonably accurate, i.e., the measurement errors $\Delta x_i \stackrel{\text{def}}{=} \tilde{x}_i - x_i$ are reasonably small. In such situations, we can expand the desired value

$$f(x_1, \dots, x_n) = f(\tilde{x}_1 - \Delta x_1, \dots, \tilde{x}_n - \Delta x_n) \quad (4)$$

in Taylor series in terms of Δx_i and keep only linear terms in this expansion; see, e.g., [5, 11, 25]. This way, we get the following approximate formula:

$$f(x_1, \dots, x_n) = f(\tilde{x}_1 - \Delta x_1, \dots, \tilde{x}_n - \Delta x_n) \approx f(\tilde{x}_1, \dots, \tilde{x}_n) - \sum_{i=1}^n \frac{\partial f}{\partial x_i} \cdot \Delta x_i, \quad (5)$$

i.e.,

$$f(x_1, \dots, x_n) = a_0 + a_1 \cdot \Delta x_1 + \dots + a_n \cdot \Delta x_n, \quad (6)$$

where we denoted $a_0 \stackrel{\text{def}}{=} f(\tilde{x}_1, \dots, \tilde{x}_n)$ and

$$a_i \stackrel{\text{def}}{=} -\frac{\partial f}{\partial x_i}(\tilde{x}_1, \dots, \tilde{x}_n). \quad (7)$$

Substituting $\Delta x_i = \tilde{x}_i - x_i$ into this formula, we conclude that

$$f(x_1, \dots, x_n) \approx c_0 + c_1 \cdot x_1 + \dots + c_n \cdot x_n, \quad (8)$$

where $c_0 = a_0 + \sum_{i=1}^n a_i \cdot \tilde{x}_i$ and $c_i = -a_i$.

Thus, from the practical viewpoint, it is important to study the case when the function $f(x_1, \dots, x_n)$ is a linear combination of its inputs.

Need for bi-intervals. In some cases, the set of possible value of a physical quantity is a union of two intervals; we will call such unions *bi-intervals*. For example, if we measured the absolute value of the velocity of an object moving along a line, and the result is $[1, 2]$, but we do not know the direction of the motion, then all we know about the actual velocity is that its value is in the union $[-2, -1] \cup [1, 2]$.

Comment. We consider situations in which we have two nested intervals:

- the larger interval \mathbf{A} – in our case, the interval $[-2, 2]$ – that contains all possible values of the corresponding quantity, and
- the smaller interval \mathbf{a} (a subset of the larger interval \mathbf{A}) – in our case, the interval $(-1, 1)$ – which *does not* contain any possible value of this quantity.

To avoid possible confusion, it should be mentioned that there are other situations in which the set S of possible values of a quantity is characterized by two nested intervals $\mathbf{a} \subseteq \mathbf{A}$. For example, in many practical situations, we know:

- a larger interval \mathbf{A} that contains the unknown set S : $S \subseteq \mathbf{A}$; this set is known as an *enclosure*, or *outer approximation*, and
- a smaller interval \mathbf{a} that is contained in the unknown set S : $\mathbf{a} \subseteq S$; this set is known as an *inner approximation* to the set s ; see, e.g., [3, 6, 7, 9, 13, 14, 16, 19, 22, 23].

This situation is different from the one that we consider in this paper. Indeed, in both cases, the relation between the set S and the larger interval \mathbf{A} is the same: we have $S \subseteq \mathbf{A}$. However, the relation between the set S and the smaller interval is different:

- in our case, the sets S and \mathbf{a} have no common elements: $S \cup \mathbf{a} = \emptyset$; while
- in the case when the interval \mathbf{a} is an inner approximation, we have $\mathbf{a} \subseteq S$.

Multi-intervals. In some cases, there are other constrains, so, in general, the set of possible values is the union of two or intervals; see, e.g., [1, 2, 4, 8, 17, 21, 26]

Comment. In this paper, multi-intervals (in particular, bi-intervals) will be denoted by capital letters A, B, \dots

How to process data until multi-interval uncertainty. If we know inputs x_i with multi-interval uncertainty, how can we compute the corresponding range of possible values of $y = f(x_1, \dots, x_n)$? Similarly to the case of interval computation, it is reasonable to first start with the case when the corresponding function $f(x_1, x_2)$ is one of the elementary arithmetic operations.

In this paper, we analyze the simplest non-interval case when all multi-intervals are bi-intervals. We start with the simplest elementary arithmetic operation – addition.

Already for the simplest arithmetic operation – addition – in general, the set of bi-intervals is not closed under addition. It is easy to come with an example when the sum of two bi-intervals is not a bi-interval: e.g., it is easy to check that

$$([0, 1] \cup [5, 6]) + ([0, 1] \cup [5, 6]) = [0, 2] \cup [5, 7] \cup [10, 12]. \tag{9}$$

A natural question. A natural question is: when is a class of bi-intervals closed under addition? under linear combination? under other operations?

In this paper, we provide answers to these questions.

2 Closeness Under Addition: Definitions and Results

Definition 1. For an interval $\mathbf{a} = [\underline{a}, \bar{a}]$, its width is defined as $w(\mathbf{a}) = \bar{a} - \underline{a}$.

Definition 2. For two intervals \mathbf{a} and \mathbf{b} , the lower distance $\underline{d}(\mathbf{a}, \mathbf{b})$ is the smallest possible value of $|a - b|$ when $a \in \mathbf{a}$ and $b \in \mathbf{b}$.

Comments.

- If the intervals $\mathbf{a} = [\underline{a}, \bar{a}]$ and $\mathbf{b} = [\underline{b}, \bar{b}]$ intersect, then clearly the lower distance is 0.
- If the intervals \mathbf{a} and \mathbf{b} are disjoint, then, without losing generality, we can assume that $\bar{a} < \underline{b}$. Then $\underline{d}(\mathbf{a}, \mathbf{b}) = \underline{b} - \bar{a}$.
- It is worth mentioning that the lower distance is different from the known Hausdorff distance

$$d_H(A, B) \stackrel{\text{def}}{=} \inf\{r > 0 : (\forall a \in A \exists b \in B (d(a, b) \leq r)) \& (\forall b \in B \exists a \in A (d(a, b) \leq r))\}. \quad (10)$$

Indeed, for two closed bounded sets A and B , the Hausdorff distance is equal to 0 if and only if the sets A and B coincide. In contrast, the lower distance can be 0 when the intervals \mathbf{a} and \mathbf{b} are different but have a common point: for example, $d_H([0, 1], [1, 2]) = 1$ but $\underline{d}([0, 1], [1, 2]) = 0$.

Definition 3. By a bi-interval, we mean either an interval or a union of two disjoint intervals $[\underline{a}, \bar{a}] \cup [\underline{b}, \bar{b}]$.

Definition 4. Let $f(x_1, \dots, x_n)$ be a function $\mathbb{R}^n \rightarrow \mathbb{R}$, and let X_1, \dots, X_n be bi-intervals. By the result $f(X_1, \dots, X_n)$ of applying the function $f(x_1, \dots, x_n)$ to bi-intervals X_i , we mean the range

$$f(X_1, \dots, X_n) \stackrel{\text{def}}{=} \{f(x_1, \dots, x_n) : x_1 \in X_1, \dots, x_n \in X_n\}. \quad (11)$$

Comment. In particular:

- When $f(x_1, x_2) = x_1 + x_2$, we will call the set $f(X_1, X_2)$ the *sum* of the bi-intervals X_i , and denote this set by $X_1 + X_2$.
- When $f(x_1, \dots, x_n) = c_0 + c_1 \cdot x_1 + \dots + c_n \cdot x_n$ for some constants c_i , we will call the set $f(X_1, \dots, X_n)$ the *linear combination* of the bi-intervals X_i , and denote this set by $c_0 + c_1 \cdot X_1 + \dots + c_n \cdot X_n$.

Definition 5. We say that a bi-interval is *close* if it is either an interval, or the union $\mathbf{a} \cup \mathbf{b}$ of two disjoint intervals for which $\underline{d}(\mathbf{a}, \mathbf{b}) \leq \max(w(\mathbf{a}), w(\mathbf{b}))$.

Comments. Since every infinite or semi-infinite interval has infinite width, this implies that every bi-interval in which one of the intervals is infinite or semi-infinite is close. In particular, every interval of the type $(-\infty, \bar{a}] \cup [\underline{b}, \infty)$ is close.

Proposition 1. A bi-interval $\mathbf{a} \cup \mathbf{b}$ is close if and only if its sum with itself is also a bi-interval.

Comment. For reader's convenience, all the proofs are placed in the special Proofs section.

Proposition 2. The sum of two close bi-intervals is always close.

These two results lead to the following characterization of all classes of bi-intervals which are closed under addition:

Theorem 1. Let \mathcal{C} be a class of bi-intervals which is closed under addition. Then every bi-interval from the class \mathcal{C} is close.

Theorem 2. The class of all close bi-intervals is closed under addition.

3 Closeness Under Linear Combination

Proposition 3. For each close bi-interval $A = \mathbf{a} \cup \mathbf{b}$ and for all c_0 and c_1 , the bi-interval

$$c_0 + c_1 \cdot A = (c_0 + c_1 \cdot \mathbf{a}) \cup (c_0 + c_1 \cdot \mathbf{b}) \tag{12}$$

is also close.

Together with closeness under addition, this result leads to the following characterization of all classes of bi-intervals which are closed under linear combination.

Theorem 3. Let \mathcal{C} be a class of bi-intervals which is closed under linear combination. Then every bi-interval from the class \mathcal{C} is close.

Theorem 4. The class of all close bi-intervals is closed under linear combination.

4 What About Other Operations

The following result shows that linear combinations are the only operations that preserve closeness of bi-intervals.

Theorem 5. For a continuously differentiable function $f(x_1, \dots, x_n)$, the following two conditions are equivalent:

- for any close bi-intervals X_1, \dots, X_n , the set $f(X_1, \dots, X_n)$ is also a close bi-interval;
- the function $f(x_1, \dots, x_n)$ is linear, i.e.,

$$f(x_1, \dots, x_n) = c_0 + c_1 \cdot x_1 + \dots + c_n \cdot x_n \tag{13}$$

for some constants c_i .

5 Proofs

Proof of Proposition 1. Let us prove that a bi-interval $\mathbf{a} \cup \mathbf{b}$ is close if and only if its sum with itself is also a bi-interval.

If the bi-interval is an interval, then its sum with itself is also an interval hence a bi-interval. So, it is sufficient to consider the case when the intervals $\mathbf{a} = [\underline{a}, \bar{a}]$ and $\mathbf{b} = [\underline{b}, \bar{b}]$ are disjoint. In this case, without losing generality, we can assume that $\bar{a} < \underline{b}$.

The sum S of the bi-interval with itself has the following form (in which we sorted the three component intervals by their lower bounds):

$$[2\underline{a}, 2\bar{a}] \cup [\underline{a} + \underline{b}, \bar{a} + \bar{b}] \cup [2\underline{b}, 2\bar{b}]. \tag{14}$$

If

$$2\bar{a} < \underline{a} + \underline{b} \tag{15}$$

and

$$\bar{a} + \bar{b} < 2\underline{b}, \tag{16}$$

then this sum is a union of three disjoint intervals, i.e., not a bi-interval; otherwise, it is a union of two (or one) intervals, i.e., a bi-interval.

The inequality (15) is equivalent to $\bar{a} - \underline{a} < \underline{b} - \bar{a}$, i.e., to $w(\mathbf{a}) < \underline{d}(\mathbf{a}, \mathbf{b})$. Similarly, the inequality (16) is equivalent to $\bar{b} - \underline{b} < \underline{b} - \bar{a}$, i.e., to $w(\mathbf{b}) < \underline{d}(\mathbf{a}, \mathbf{b})$. Thus, both inequalities are satisfied if and only if $\max(w(\mathbf{a}), w(\mathbf{b})) < \underline{d}(\mathbf{a}, \mathbf{b})$, i.e., exactly if and only if the bi-interval is *not* close.

The proposition is proven.

Proof of Proposition 2. Let us prove that the sum of two close bi-intervals is always close.

Indeed, when we add intervals, their width increases (or at least not decreases), while the lower distance only decreases (or at least remains the same), thus the inequality remains. The proposition is proven.

Proof of Proposition 3. Let us prove that for each close bi-interval $\mathbf{a} \cup \mathbf{b}$ and for all c_0 and c_1 , the bi-interval $(c_0 + c_1 \cdot \mathbf{a}) \cup (c_0 + c_1 \cdot \mathbf{b})$ is also close.

Indeed, a shift by c_0 does not change the lower distance and the widths, and multiplication by c_1 multiplies all these values by $|c_1|$. Thus, for the new intervals, the inequality describing closeness still remains.

The proposition is proven.

Proof of Theorem 5.

1°. We have already proven that a linear combination of close bi-intervals is a close bi-interval. So, to prove our theorem, it is sufficient to prove that if some continuously differentiable function $f(x_1, \dots, x_n)$ always transforms close bi-intervals into a close bi-interval, this means that this function is linear.

So let us assume that $f(x_1, \dots, x_n)$ is such a function, and let us prove that this function is linear.

2°. Let us start with the case when $n = 1$, i.e., when $f(x_1)$ is a continuously differentiable function of one variable.

If the derivative $f'(x_1)$ is always equal to 0, this means that this function is a constant – and is, therefore, a linear function. Let us now consider the case when the function is not constant, i.e., when there exist values x_1 for which $f'(x_1) \neq 0$. Let us pick any such value x_0 . Without losing generality, let us assume that $f'(x_0) > 0$; the case when $f'(x_0) < 0$ can be treated the same way.

Since the derivative $f'(x_1)$ is continuous, from the fact that $f'(x_0) > 0$ it follows that the derivative is positive in some open neighborhood N of the point x_0 . So, in this neighborhood N , the function $f(x_1)$ is strictly increasing. For an increasing function $f(x_1)$, its range $f([\underline{x}, \bar{x}])$ on any interval is simply equal to $[f(\underline{x}), f(\bar{x})]$.

For any three points $x - h$, x , and $x + h$ from the neighborhood N , the bi-intervals $[x - h, x - h] \cup [x, x + h]$ and $[x - h, x] \cup [x + h, x + h]$ are close. The image

$$f([x - h, x - h] \cup [x, x + h]) \quad (17)$$

of the first close bi-interval is thus equal to $[f(x - h), f(x - h)] \cup [f(x), f(x + h)]$. This is clearly a bi-interval. The fact that this bi-interval is close means that

$$f(x) - f(x - h) \leq f(x + h) - f(x). \quad (18)$$

The fact that the image

$$f([x - h, x] \cup [x + h, x + h]) = [f(x - h, f(x)), [f(x + h, f(x + h))] \quad (19)$$

is close means that

$$f(x + h) - f(x) \leq f(x) - f(x - h). \quad (20)$$

From the inequalities (18) and (20), we conclude that

$$f(x+h) - f(x) = f(x) - f(x-h). \quad (21)$$

In particular, this means that for any value a and for all k for which $a+k \cdot h \in N$, we have

$$\begin{aligned} f(a+h) - f(a) &= f(a+2h) - f(a+h) = \dots = \\ &= f(a+k \cdot h) - f(a+(k-1) \cdot h). \end{aligned} \quad (22)$$

If we denote this common difference by $d \stackrel{\text{def}}{=} f(a+h) - f(a)$, we conclude that $f(a+i \cdot h) = f(a) + i \cdot d$, i.e., that $f(a+\Delta) = f(a) + (d/h) \cdot \Delta$ for all values Δ of the type $\Delta = i \cdot h$. In the limit $h \rightarrow 0$, we conclude that the function $f(x_1)$ is linear in the neighborhood N . Thus, in this neighborhood, the derivative $f'(x_1)$ is constant.

This is true for every neighborhood in which the derivative is positive. Since the derivative is continuous, it cannot jump to 0 or to a negative number, so the derivative will be everywhere positive and thus, everywhere constant. So, the function $f(x_1)$ is indeed linear.

3°. Let us now consider the case when $n = 2$, i.e., when $f(x_1, x_2)$ is a function of two variables. By taking $X_2 = [x_2, x_2]$, we conclude, from Part 2 of this proof, that for every x_2 , the function $x_1 \rightarrow f(x_1, x_2)$ is linear, i.e., that

$$f(x_1, x_2) = c_0(x_2) + c_1(x_2) \cdot x_1 \quad (23)$$

for some coefficients $c_0(x_2)$ and $c_1(x_2)$ which are, in general, different for different values x_2 . Similarly, we can conclude that

$$f(x_1, x_2) = a_0(x_1) + a_1(x_1) \cdot x_2. \quad (24)$$

Equating the two expressions for $f(x_1, x_2)$, we conclude that for all x_1 and x_2 , we have

$$c_0(x_2) + c_1(x_2) \cdot x_1 = a_0(x_1) + a_1(x_1) \cdot x_2. \quad (25)$$

Let us consider two possible cases:

- when the function $c_1(x_2)$ is constant, and
- when this function is not constant.

3.1°. If the function $c_1(x_2)$ is constant, i.e., if $c_1(x_2) = c_1$ for all x_2 , then from (25) for $x_1 = 0$, we conclude that

$$c_0(x_2) = a_0(0) + a_1(0) \cdot x_2. \quad (26)$$

Substituting this expression for $c_0(x_2)$ and $c_1(x_2) = c_1$ into the formula (N3), we conclude that

$$f(x_1, x_2) = a_0(0) + a_1(0) \cdot x_2 + c_1 \cdot x_1, \quad (27)$$

i.e., that $f(x_1, x_2)$ is a linear function of two variables.

3.2°. Let us now consider the case when the function $c_1(x_2)$ is not a constant, i.e., when there exist two values a_2 and b_2 for which $c_1(a_2) \neq c_1(b_2)$. Substituting $x_2 = a_2$ and $x_2 = b_2$ into the formula (25), we get the following two equalities:

$$c_0(a_2) + c_1(a_2) \cdot x_1 = a_0(x_1) + a_1(x_1) \cdot a_2; \quad (28)$$

$$c_0(b_2) + c_1(b_2) \cdot x_1 = a_0(x_1) + a_1(x_1) \cdot b_2. \quad (29)$$

We can view this as a system of two linear equations with two unknowns $a_0(x_1)$ and $a_1(x_1)$. Subtracting these two equations, we conclude that

$$a_1(x_1) \cdot (a_2 - b_2) = (c_0(a_2) - c_0(b_2)) + (c_1(a_2) - c_1(b_2)) \cdot x_1, \quad (30)$$

hence, that

$$a_1(x_1) = \frac{c_0(a_2) - c_0(b_2)}{a_2 - b_2} + \frac{c_1(a_2) - c_1(b_2)}{a_2 - b_2} \cdot x_1. \quad (31)$$

So, $a_1(x_1)$ is a linear function of x_1 . Now, from (28), we conclude that

$$a_0(x_1) = c_0(a_2) + c_1(a_2) \cdot x_1 - a_1(x_1) \cdot a_2, \quad (32)$$

and thus, that $a_0(x_1)$ is also a linear function.

Substituting the linear expressions for $a_0(x_1)$ and $a_1(x_1)$ into the expression (N4), we conclude that the function $f(x_1, \dots, x_n)$ is *bilinear*, i.e., has the form

$$f(x_1, x_2) = f_0 + f_1 \cdot x_1 + f_2 \cdot x_2 + f_{12} \cdot x_1 \cdot x_2, \quad (33)$$

for some constant f_i .

4°. Similarly, we can prove that for all n , the function $f(x_1, \dots, x_n)$ is *multi-linear*, i.e., has the form

$$f(x_1, \dots, x_n) = f_0 + \sum_{i=1}^n f_i \cdot x_i + \sum_{i_1 < i_2} f_{i_1 i_2} \cdot x_{i_1} \cdot x_{i_2} + \sum_{i_1 < i_2 < i_3} f_{i_1 i_2 i_3} \cdot x_{i_1} \cdot x_{i_2} \cdot x_{i_3} + \dots + f_{1\dots n} \cdot x_1 \cdot \dots \cdot x_n. \quad (34)$$

5°. Let us prove that for $n = 2$, the function $f(x_1, x_2)$ must be linear. We will prove it by contradiction.

Indeed, let us assume that the function $f(x_1, x_2)$ is *not* linear, i.e., that it has the form (33) with $f_{12} \neq 0$. Our assumption about this function is that for all close bi-intervals X_1 and X_2 , the range $f(X_1, X_2)$ is also a close bi-interval. Since a multiplication by a constant does not change closeness, this implies that the intervals $g(X_1, X_2) = f_{12}^{-1} \cdot f(X_1, X_2)$ is also close, where

$$g(x_1, x_2) \stackrel{\text{def}}{=} f_{12}^{-1} \cdot f(x_1, x_2) = g_0 + g_1 \cdot x_1 + g_2 \cdot x_2 + x_1 \cdot x_2, \quad (35)$$

and $g_i \stackrel{\text{def}}{=} f_{12}^{-1} \cdot f_i$. The expression for $g(x_1, x_2)$ can be rewritten as

$$g(x_1, x_2) = (x_1 + g_2) \cdot (x_2 + g_1) + (g_0 - g_1 \cdot g_2). \quad (36)$$

Since adding a constant does not change closeness, this implies that the set

$$(X_1 + g_2) \cdot (X_2 + g_1) \quad (37)$$

is also a close bi-interval.

For any two close bi-intervals Y_1 and Y_2 , the bi-intervals $X_1 = Y_1 - g_2$ and $X_2 = Y_2 - g_1$ are also close. Thus, the range

$$(X_1 + g_2) \cdot (X_2 + g_1) = ((Y_1 - g_2) + g_2) \cdot ((Y_2 - g_1) + g_1) = Y_1 \cdot Y_2 \quad (38)$$

is also a close bi-interval. So, we conclude that the product of two close bi-intervals should also be a close bi-interval.

But this is not always true: e.g., for close bi-intervals $Y_1 = Y_2 = [-0.4, -0.4] \cup [0.6, 1.6]$, the product $Y_1 \cdot Y_2$ is not a bi-interval: it is the union of three disjoint intervals:

$$Y_1 \cdot Y_2 = [-0.64, -0.24] \cup [0.16, 0.16] \cup [0.36, 2.56]. \quad (39)$$

The contradiction shows that we cannot have $f_{12} \neq 0$. So, $f_{12} = 0$, and the function $f(x_1, x_2)$ is linear.

6°. Similarly to Part 5 of this proof, we can conclude that in the multi-linear function $f(x_1, \dots, x_n)$, all the coefficients $f_{i_1 i_2}$ should be equal to 0.

Let us show that all the terms $f_{i_1 i_2 i_3}$ should also be equal to 0. Indeed, if $f_{i_1 i_2 i_3} \neq 0$ for some i_j , then substituting $x_{i_3} = 1$ and $x_i = 0$ for all i different from i_j into the general expression for a multi-linear function, we get a function of two variables

$$f(x_{i_1}, x_{i_3}) = f_0 + f_{i_1} \cdot x_{i_1} + f_2 \cdot x_{i_2} + f_{i_3} + f_{i_1 i_2 i_3} \cdot x_{i_1} \cdot x_{i_2}. \quad (40)$$

This function of two variables should also transform close bi-intervals into close bi-intervals, but we have already shown that, when we have a non-zero coefficient at the product $x_{i_1} \cdot x_{i_2}$, this is not possible.

Similarly, we can prove that all other non-linear coefficients $f_{i_1 \dots i_k}$ should also be equal to 0, and thus, that the function $f(x_1, \dots, x_n)$ is indeed linear. The theorem is proven.

6 Conclusions and Remaining Open Questions

Conclusions. In data processing, it is important to analyze how uncertainty in the inputs affects the results of data processing. In the case of interval uncertainty, this analysis is simplified by the fact that the sum, difference, and product of two intervals is also an interval. In the linearized case, what is important is that the linear combination of intervals is also an interval.

In some practical situations, the uncertainty in some (or all) inputs is characterized not by a *single* interval, but by a *union* of two (or more) intervals. Such unions are known as *multi-intervals*, or, in the same of a union of two intervals, *bi-intervals*. It is known that already the set of bi-intervals is not closed under addition. A natural question is: what classes of bi-intervals are closed under addition? under linear combination? under other operations?

In this paper, we give a full characterization of all classes of bi-intervals which are closed under addition and under linear combination. Namely, for each corresponding bi-interval, the gap between the two united intervals cannot exceed the largest width of the two intervals. We can such bi-intervals *close*. We also show that the class of all close bi-intervals is *not* closed under any non-linear operation (such as multiplication).

Remaining open questions.

- We proved that the only operations preserving the class of all close bi-intervals are linear ones. A natural question is: is there a subclass of the class of all close bi-intervals which is closed under addition and multiplication? under addition, multiplication, and inverse $1/x$?

- What if instead of bi-intervals – i.e., unions of at most two disjoint intervals – we consider *tri-intervals*, i.e., unions of no more than 3 disjoint intervals? What if we consider *n-intervals* – unions of no more than n disjoint intervals?

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