

Realistic Intervals of Degrees of Confidence*

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Abstract

One of the applications of intervals is in describing experts' degrees of confidence in their statements. In this application, not all intervals are realistically possible. To describe all realistically possible degrees of confidence, we end up with a mathematical question of describing all topologically closed classes of intervals which are closed under the appropriate minimum and maximum operations. In this paper, we provide a full description of all such classes.

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1 Formulation of the Problem

Numerical and interval-valued fuzzy degrees: a brief reminder. One of the main ideas behind traditional logic is that every statement is either true or false. In the computer, “true” is usually represented as 1 and “false” as 0.

In practice, people are often not 100% confident in their statements. A natural way to describe a person's degree of confidence in a statement is to use numbers intermediate between the value 1 (corresponding to full confidence) and the value 0 (corresponding to the complete absence of confidence). This representation of degrees of confidence by numbers from the interval $[0, 1]$ is one of the main ideas behind *fuzzy logic*, a methodology for transforming imprecise (“fuzzy”) expert statements into precise computer-understandable form; see, e.g., [1, 4, 8, 10, 11, 12].

If we have two statements A_1 and A_2 with degrees of confidence a_1 and a_2 , then our degree of confidence in a composite statement $A_1 \wedge A_2$ (A_1 and A_2) cannot exceed neither a_1 nor a_2 and thus, must be smaller than or equal to $\min(a_1, a_2)$. Similarly, our degree of confidence in a composite statement $A_1 \vee A_2$ (A_1 or A_2) cannot be smaller than either a_1 or a_2 and thus, must be greater than or equal to $\max(a_1, a_2)$.

The most natural way to estimate the experts' degrees of confidence is to ask the experts themselves. The problem is that just like the expert is not certain about

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his/her statement, he/she is also not confident about the exact degree of confidence. From this viewpoint, it is more natural to characterize the degree of confidence not by a single value, but by an *interval* of possible values, i.e., by a subinterval of the interval $[0, 1]$; see, e.g., [8].

The minimum and maximum operations can be naturally extended to such intervals, in the usual interval-computation way (see, e.g., [2, 7, 9]):

$$\min([a_1, b_1], [a_2, b_2]) \stackrel{\text{def}}{=} \{\min(v_1, v_2) : v_1 \in [a_1, b_1] \text{ and } v_2 \in [a_2, b_2]\};$$

$$\max([a_1, b_1], [a_2, b_2]) \stackrel{\text{def}}{=} \{\max(v_1, v_2) : v_1 \in [a_1, b_1] \text{ and } v_2 \in [a_2, b_2]\}.$$

Since both min and max are monotonic functions, these ranges are easy to compute exactly:

$$\min([a_1, b_1], [a_2, b_2]) = [\min(a_1, a_2), \min(b_1, b_2)]; \quad (1)$$

$$\max([a_1, b_1], [a_2, b_2]) = [\max(a_1, a_2), \max(b_1, b_2)]. \quad (2)$$

Not all intervals are realistic. In principle, we can consider all possible subintervals of the interval $[0, 1]$, but in practice, not all of them appear. For example, the interval $[0, 1]$ would mean that the expert has no idea whether his/her statement is true or not – but in this case, the expert would not make this statement.

Similarly, the interval $[0.1, 0.9]$ is not very realistic. This simply means that;

- a person has some very small amount of positive knowledge – preventing the expert from assigning degrees of confidence below 0.1, and
- a person has some very small amount of negative knowledge – preventing the expert from assigning degrees of confidence larger than 0.9.

This interval is reasonable for a person in the street, who has very little knowledge about the situation, but we do not expect this, in effect, “I don’t know” answer from someone who we consider to be an expert.

It is desirable to describe the class of all realistic intervals. Traditionally, when we design algorithms for processing expert opinions, we do not impose any restrictions on possible intervals – these algorithms can process both:

- realistic intervals, i.e., intervals that an expert can produce, and
- unrealistic intervals, i.e., intervals which are mathematically possible but which we do not expect to come from an expert.

Maybe if we can describe, in precise terms, which intervals are realistic and which are not, this can help us avoid wasting time on thinking how to process unrealistic intervals and concentrate on designing algorithms focused on realistic intervals only.

Examples. Let us start with some natural ideas on how the class of realistic intervals can look like.

Based on the above argument, an interval is unreasonable if it is too wide. So, a natural idea is to set up the maximum possible width δ of such an interval and only consider an interval $[a, b]$ realistic if $b \leq a + \delta$.

Another option is to take into account that, according to empirical studies, the degrees of confidence provided by an expert (which are also known as *subjective probabilities*) are related to actual (objective) probabilities by a nonlinear dependence: for example, we routinely overestimate the probability of rare events; see, e.g., [3, 5, 6]. In view of this, instead of considering the width of the original interval $[a, b]$, maybe a good idea is:

- first, to transform this interval into probabilities by using an appropriate non-linear transformation $g(x)$, and
- then, to consider the width of the resulting interval $[g(a), g(b)]$.

In this approach, we consider an interval $[a, b]$ realistic if $g(b) \leq g(a) + \delta$, for some constant $\delta > 0$.

There may be other realistic classes.

Natural properties of a class of realistic intervals. To describe all possible classes of realistic intervals, let us brainstorm on what are the natural properties of such a class.

First, it is reasonable to require that this class be closed under the application of operations (1) and (2) corresponding to “and” and “or”. If an expert has some realistic interval of possible degree of confidence for a statement S_1 , and another realistic interval for a statement S_2 , then for the composite statements “ S_1 and S_2 ” and “ S_1 or S_2 ” we expect the intervals obtained by using formulas (1) and (2).

Another feature is related to the fact that close values of degree of confidence are practically indistinguishable. We can meaningfully distinguish between, e.g., degree of confidence 0.6 and 0.8 – which correspond to 3 and 4 on the scale from 0 to 5. We can most probably meaningfully distinguish between degrees of confidence 0.7 and 0.8 – which correspond to 7 and 8 on the scale from 0 to 10. However, it is highly improbable that someone can meaningfully distinguish between marks 80 and 81 on a scale from 0 to 100 – i.e., between values 0.80 and 0.81. As a result, when we have a sequence of realistic intervals that converges to a limit, this limit is indistinguishable from all sufficiently close elements of this sequence – so it makes sense to consider this limit interval realistic too. In other words, it makes sense to require that the class of realistic intervals be topologically closed.

In this paper, we describe all resulting classes. The result is simple, and the proof is straightforward.

2 Result

Definition. *By a class of realistic intervals, we mean the class \mathcal{C} of subintervals of the interval $[0, 1]$ which satisfies the following three conditions:*

- *the class \mathcal{C} contains all numbers from this interval i.e., all degenerate intervals $[a, a]$,*
- *the class \mathcal{C} is topologically closed, i.e., if this class contains a sequence of intervals $[a_n, b_n]$ for which $a_n \rightarrow a$ and $b_n \rightarrow b$, then this class also contains the limit interval $[a, b]$, and*
- *the class \mathcal{C} is closed under minimum and maximum operations (1) and (2).*

Theorem. *Let f be a monotonic right-continuous function from $[0, 1]$ to $[0, 1]$ such that $a \leq f(a)$ for all a in $[0, 1]$. Then the class*

$$\mathcal{C}_f \stackrel{\text{def}}{=} \{[a, b] : a \leq b \leq f(a)\} \quad (3)$$

is a class of realistic intervals. Conversely, every class of realistic intervals occurs in this way.

Reminder. Right-continuous means that for every sequence $a_n \rightarrow a$ for which $a_n \geq a$ for all n , we have $f(a_n) \rightarrow f(a)$.

Comments.

- Right continuity is clearly *not* the most intuitively convincing concept. The only reason why we mention right continuity in the formulation of the theorem is that for a class \mathcal{C} of realistic intervals to satisfy the three natural (and hopefully intuitively convincing) conditions – as described in the Definition – is to have a function f that is right continuous.
- In addition to describing the interval of possible values of the expert’s degrees of confidence, a natural idea is to also assign, to each possible value v of the degree of confidence, a number $\mu(v)$ from the interval $[0, 1]$ indicating to what extent the expert believes that this value v adequately described his/her opinion. This construction – known as *type-2 fuzzy set* (see, e.g., [8]) – generalizes the interval-valued degrees of confidence as described in this paper: namely, the interval-valued degrees of confidence correspond to the case when:
 - for all values v from the corresponding interval, we have $\mu(v) = 1$, while
 - for all values outside this interval, we have $\mu(v) = 0$.

Operations of minimum and maximum can be naturally extended to such type-2 fuzzy sets. It is therefore desirable to extend our result to such type-2 sets – i.e., to analyze which classes of type-2 sets are closed under similar operations.

Examples.

- The above-described class of all intervals of width not exceeding some threshold $\delta > 0$ can be described as the class \mathcal{C}_f for the function $f(a) = \min(a + \delta, 1)$.
- The second above-described class – corresponding to width after a nonlinear rescaling – corresponds to the function $f(a) = g^{-1}(\min(g(a) + \delta, 1))$.

Proof of the Theorem.

1°. Let us first show that for every monotonic right-continuous function $f(a)$ for which $a \leq f(a)$ for all a , the formula (3) defines a class of realistic intervals, i.e., that the resulting class \mathcal{C}_f :

- contains all numbers from this interval,
- is topologically closed, and
- is closed under minimum and maximum operations.

Let us prove these properties one by one.

1.1°. Since $a \leq f(a)$, each interval $[a, a]$ belongs to the class \mathcal{C}_f .

1.2°. Let us prove that the class \mathcal{C}_f is topologically closed, i.e., that if $[a_n, b_n] \rightarrow [a, b]$ and $[a_n, b_n] \in \mathcal{C}_f$ for all n , then $[a, b] \in \mathcal{C}_f$.

By definition of the class (3), for each n , we have $b_n \leq f(a_n)$. Let us consider two cases.

1.2.1°. If for infinitely many values a_{n_k} , we have $a_{n_k} \leq a$, then, due to monotonicity, we have $f(a_{n_k}) \leq f(a)$, so for $b_{n_k} \leq f(a_{n_k})$, we also have $b_{n_k} \leq f(a)$, and in the limit, we get $b \leq f(a)$, i.e., $[a, b] \in \mathcal{C}_f$.

1.2.2°. In the opposite case, when for all but finitely many indices n , we have $a < a_n$, then, due to right continuity, we have $f(a_n) \rightarrow f(a)$. Thus, from $b_n \leq f(a_n)$, in the limit, we get $b \leq f(a)$, i.e., also $[a, b] \in \mathcal{C}_f$.

1.3°. Let us show that the class \mathcal{C}_f is closed under the minimum operation.

Indeed, let us assume that $[a_1, b_1] \in \mathcal{C}_f$ and $[a_2, b_2] \in \mathcal{C}_f$, i.e., $b_1 \leq f(a_1)$ and $b_2 \leq f(a_2)$. Let us prove that for $[a, b] \stackrel{\text{def}}{=} [\min(a_1, a_2), \min(b_1, b_2)]$, we also have $[a, b] \in \mathcal{C}_f$. Without losing generality, we can assume that $a_1 \leq a_2$, so $a = \min(a_1, a_2) = a_1$. In this case, $b = \min(b_1, b_2) \leq b_1 \leq f(a_1) = f(a)$, so indeed $[a, b] \in \mathcal{C}_f$.

1.4°. Finally, let us show that the class \mathcal{C}_f is closed under the maximum operation.

Indeed, let us assume that $[a_1, b_1] \in \mathcal{C}_f$ and $[a_2, b_2] \in \mathcal{C}_f$, i.e., $b_1 \leq f(a_1)$ and $b_2 \leq f(a_2)$. Let us prove that for $[a, b] \stackrel{\text{def}}{=} [\max(a_1, a_2), \max(b_1, b_2)]$, we also have $[a, b] \in \mathcal{C}_f$. Without losing generality, we can assume that $a_1 \leq a_2$, so $a = \max(a_1, a_2) = a_2$. Here, $b_2 \leq f(a_2)$, and since $b_1 \leq f(a_1)$ and $f(a)$ is a monotonic function, $f(a_1) \leq f(a_2)$ and thus, $b_1 \leq f(a_2)$. From $b_1 \leq f(a_2)$ and $b_2 \leq f(a_2)$, we conclude that $b = \max(b_1, b_2) \leq f(a_2) = f(a)$, so indeed $[a, b] \in \mathcal{C}_f$.

2°. Let \mathcal{C} be a class of realistic intervals. Let us prove that this class is equal to \mathcal{C}_f for some function $f(a)$.

Indeed, for each a , there are intervals $[a, b]$ in the class \mathcal{C} – e.g., the degenerate interval $[a, a]$. Let us define

$$f(a) \stackrel{\text{def}}{=} \sup\{b : [a, b] \in \mathcal{C}\}. \quad (4)$$

The supremum $f(a)$ is a limit of values b_n for which $[a, b_n] \in \mathcal{C}$. Thus, since the class \mathcal{C} is topologically closed, the interval $[a, f(a)]$ also belongs to this class \mathcal{C} :

$$[a, f(a)] \in \mathcal{C}. \quad (5)$$

3°. By definition of the supremum, no interval $[a, b]$ with $b > f(a)$ belongs to the class \mathcal{C} . On the other hand, if $a \leq b \leq f(a)$, then the interval $[a, b]$ indeed belongs to the class \mathcal{C} , since this interval is equal to the result $\min([b, b], [a, f(a)])$ of applying the operation (1) to intervals $[b, b]$ and $[a, f(a)]$ from this class – and the class \mathcal{C} is closed under this operation.

Thus, the class \mathcal{C} indeed has the form (3).

4°. Let us prove that the function $f(a)$ defined by the formula (4) is indeed monotonic, right-continuous, and satisfies the condition $a \leq f(a)$.

4.1°. The last condition is the easiest to prove, since the class \mathcal{C} contain an interval $[a, f(a)]$, and the upper endpoint of an interval is always great than or equal to its lower endpoint.

4.2°. Monotonicity is also easy: if $a \leq a'$, then, since $[a, f(a)] \in \mathcal{C}$, we have

$$\max([a', a'], [a, f(a)]) = [a', \max(a', f(a))] \in \mathcal{C}.$$

Thus, by definition of $f(a')$ as the largest b for which $[a', b] \in \mathcal{C}$, we conclude that $\max(a', f(a)) \leq f(a')$. Since $f(a) \leq \max(a', f(a))$, we thus get $f(a) \leq f(a')$.

4.3°. Let us now prove right continuity. Suppose that $a_n \rightarrow a$ and $a_n \geq a$ for all n . Since the function $f(a)$ is monotonic, we have $f(a) \leq f(a_n)$ for all n and thus, in the limit:

$$f(a) \leq \liminf f(a_n). \quad (6)$$

Due to (3), for each n , we have $[a_n, f(a_n)] \in \mathcal{C}$. For a subsequence $f(a_{n_k})$ that converges to the upper limit $\limsup f(a_n)$, due to topological closeness, we have $[a, \limsup f(a_n)] \in \mathcal{C}$. By definition (4) of the function $f(a)$, this means that

$$\limsup f(a_n) \leq f(a). \quad (7)$$

Since we always have $\liminf f(a_n) \leq \limsup f(a_n)$, inequalities (6) and (7) imply that

$$f(a) \leq \liminf f(a_n) \leq \limsup f(a_n) \leq f(a),$$

thus

$$\liminf f(a_n) = \limsup f(a_n) = f(a).$$

Since the lower and upper limits coincide, this means that the sequence $f(a_n)$ indeed has a limit, and this limit is equal to $f(a)$. Right continuity is now proven.

The proposition is proven.

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