Further Matrix Classes Possessing the Interval Property*

Jürgen Garloff
Department of Mathematics and Statistics, University of Konstanz, and Institute for Applied Research, University of Applied Sciences / HTWG Konstanz, Germany
juergen.garloff@htwg-konstanz.de

Doaa Al-Saafin
Department of Mathematics and Statistics, University of Konstanz, Germany
doaa.al-saafin@uni-konstanz.de

Mohammad Adm
Department of Applied Mathematics and Physics, Palestine Polytechnic University, Hebron, Palestine
moh_95@ppu.edu

Abstract

In this article, the collection of classes of matrices which possess the interval property presented in [J. Garloff, M. Adm, and J. Titi, A survey of classes of matrices possessing the interval property and related properties, Reliab. Comput., 22:1-14, 2016] is continued. That is, given an interval of matrices with respect to a certain partial order, it is desired to know whether a special property of the entire matrix interval can be inferred from some of its element matrices lying on the vertices of the matrix interval. The interval property of some matrix classes found in the literature is presented, and the interval property of further matrix classes including the ultrametric, the conditionally positive semidefinite, and the infinitely divisible matrices is given for the first time. For the inverse M-matrices, the cardinality of the required set of vertex matrices known so far is significantly reduced.

Keywords: Interval property, matrix interval, inverse M-matrix, ultrametric matrix, conditionally positive semidefinite matrix, infinitely divisible matrix
AMS subject classifications [2021]: 15-02, 15B48, 65G99

*Submitted: January 14, 2021; Revised: June 1, 2021; Accepted: June 21, 2021.

1 Introduction

In this article, we continue the survey [18] and present further classes of matrices which possess the so-called interval property. The interval properties which are proven in our paper are presented for the first time. This applies, e.g., to the ultrametric, the conditionally positive (negative) semidefinite, and the infinitely divisible matrices, see IP 3.2, 4.7 and 4.8 below. For the inverse M-matrices the cardinality of the required set of vertex matrices known so far is significantly reduced, see IP 4.4.
Many of the classes listed below are related to the linear complementarity problem [14]. Often properties of this problem like solvability, uniqueness, convexity, and finite number of solutions are reflected by properties of the constraint matrix. For a collection of respective matrix classes see [13]. In the case that one considers the linear complementarity problem with uncertain data modeled by intervals it is important to know whether all matrices obtained by possible values in the intervals are in the same matrix class. Then it is an enormous advantage if one could infer this property from a finite set of matrices.
For the ease of presentation, we keep the notation introduced in [18]. We affix an asterisk to all references to [18], e.g., Theorem 3.2* means Theorem 3.2 in the first part.

2 Matrix Properties Which Can Be Inferred from One or Two Vertex Matrices

2.1 Matrix Intervals with Respect to the Usual Entry-wise Partial Order

We start with classes of matrices which are closely related to the linear complementarity problem, see, e.g., [13], [20]. As in Subsection 3.1*, the strict and non-strict inequalities between vectors and matrices are understood entry-wise, e.g., for $x \in R^n$, $x > 0$ and $x \geq 0$ mean $x_i > 0$ and $x_i \geq 0$, respectively, $i = 1, \ldots, n$.

Definition 2.1. *A matrix $A \in \mathbb{R}^{m,n}$ is called*

(i) semipositive*, if there exists a vector $x \in \mathbb{R}^n$ with $x \geq 0$ such that $Ax > 0$;*

(ii) minimally semipositive*, if it is semipositive and no column-deleted submatrix is semipositive;*

and if $m = n$, A is called

(iii) (strictly) semimonotone*, if each nonzero vector $x \in \mathbb{R}^n$ with $x \geq 0$ has a component $x_k > 0$ such that $(Ax)_k \geq 0$ (> 0);*

(iv) (strictly) copositive*, if $x^TAx \geq 0$ (> 0) for all vectors $x \in \mathbb{R}^n$ with $x \geq 0$ (and $x \neq 0$).*

A thorough treatment of properties of the (minimally) semipositive, the strictly semimonotone, and the (strictly) copositive matrices can be found in [25, Chapter 3, Section 3.8, and Chapter 6]. Obviously, each of the matrix classes (i), (iii), and (iv) in Definition 2.1 has the interval property, see [20, Propositions 1,3,5].

IP 2.1. *Each of the sets of*

(i) the semipositive matrices,

(ii) the (strictly) semimonotone matrices, and

(iii) the (strictly) copositive matrices

has the interval property with respect to the left corner matrix.

The minimally semipositive matrices possess the interval property with respect to the two corner matrices but a slightly stronger result is valid.

Theorem 2.1. *[12, Corollaries 5.5 and 5.6], [10, Theorem 5.1 (b)] Let* $[A] = [\underline{A}, \overline{A}]$ *be a matrix interval in* $\mathbb{R}^{m,n}$. *Then* $[A]$ *is minimally semipositive if and only if* $\underline{A}$ *is semipositive and* $\overline{A}$ *is minimally semipositive.*

It is known, see, e.g., [25, Corollary 3.5.8], that a square matrix A is minimally semipositive if and only if A is inverse nonnegative, i.e., A is nonsingular and $A^{-1} \geq 0$ (Definition 3.1*). Thus, Theorem 2.1 provides a strengthening of IP 3.1.1*. For related results, see [12, Section 5].

Given a (minimally) semipositive matrix B, results on constructing a (minimally) semipositive matrix interval $[A, B]$ can be found in [25, Section 3.7.3].

Reversing the sign of the determinant of P-matrices, i.e, of matrices having all their principal minors positive (Definition 4.5*), results in the class of the following matrices which are related to the linear complementarity problem [33].

Definition 2.2. *A matrix* $A \in \mathbb{R}^{n,n}$, $n \geq 2$, *is called an* almost P-matrix *if all its proper principal minors are positive while its determinant is negative.*

IP 2.2. *[16, Lemma 3] The set of the almost* P*-matrices having all their off-diagonal entries nonpositive has the interval property with respect to the two corner matrices.*

The almost P-matrices appearing in IP 2.2 have the property that all their proper principal submatrices are M-matrices. Equivalent to the Definition 3.1* for a matrix A to be an M-matrix is that A can be represented as

$$A = tI - B \text{ with } B \geq 0, \tag{2.1}$$

where I is the identity matrix and $t > \rho(B)$ ($\rho(B)$ denotes the spectral radius of B). A characterization of a matrix A appearing in IP 2.2 is that it can be represented as in (2.1) with $\sigma < t < \rho(B)$, where σ denotes the maximum of the spectral radii of all principal submatrices of order $n-1$ of B [16, p. 188]. In [26], the closure of these matrices is considered, i.e., the matrices A having the following properties:

(a) A can be represented as in (2.1),

(b) $\sigma \leq t < \rho(B)$,

(c) the off-diagonal entries of A are nonpositive.

IP 2.3. *[26, Theorem 2.10(i)] The set of the matrices* A *with (a)-(c) has the interval property with respect to the two corner matrices.*

The set of the M-matrices is contained in the set of the H-matrices.

Definition 2.3. *A matrix $A = (a_{ij}) \in \mathbb{R}^{n,n}$ is called an H-*matrix *if $\langle A \rangle$ an M-matrix, where $\langle A \rangle$ is defined as follows*

$$\langle A \rangle_{ij} := \begin{cases} |a_{ii}| & \text{if } i = j, \\ -|a_{ij}| & \text{if } i \neq j, \end{cases}$$

for all $i, j = 1, \ldots, n$.

Definition 2.4. *Let $[A]$ be a matrix interval in $\mathbb{R}^{n,n}$. We are making use of the representation of $[A]$ as an interval matrix, i.e., $[A] = ([a_{ij}])_{i,j=1}^{n}$, see (2)*, and define the vertex matrix $v([A])$ as follows:*

$$v([A])_{ij} := \begin{cases} \min\limits_{a_{ii} \in [a_{ii}]} |a_{ii}| & \text{if } i = j, \\ -\max\limits_{a_{ij} \in [a_{ij}]} |a_{ij}| & \text{if } i \neq j, \end{cases}$$

for all $i, j = 1, \ldots, n$.

IP 2.4. *E.g., [32, Theorem 3.3.5(a)] The set of the H-matrices has the interval property with respect to a vertex matrix at which $v([A])$ is attained.*

Sufficient conditions for a matrix interval to contain only H-matrices can be found in [7]. Also, the H-matrix intervals are related to the verified numeric solution of the linear complementarity problem with interval data, see [3] and [31] and further references therein.

2.2 Matrix Intervals with Respect to the Checkerboard Partial Order

In this section, the underlying order is the checkerboard partial order, see Section 3.2*. We start with a class of matrices which are named after the French mathematician, physicist, and chemist Gaspard Monge (1746-1818). For references and applications of these matrices, e.g., in optimization, see [6].

Definition 2.5. *A matrix $A = (a_{ij}) \in \mathbb{R}^{m,n}$ is called* Monge matrix *if for all i, j, k, l with $1 \leq i < k \leq m$ and $1 \leq j < l \leq n$, it holds that*

$$a_{ij} + a_{kl} \leq a_{il} + a_{kj}.$$

IP 2.5. *[6, Theorem 3.1] The set of the Monge matrices has the interval property with respect to the two corner matrices.*

In the remainder of this section we consider special sign regular matrices, i.e., matrices whose nonzero minors of a fixed order have the same sign, see Section 3.2*. Recently, the relation between the subclasses of the totally nonnegative (TN), totally positive (TP), totally nonpositive ($t.n.p.$), and totally negative ($t.n.$) matrices and the linear complementarity problem has been discovered [8], [9].

Definition 2.6. *A matrix $A \in \mathbb{R}^{m,n}$ is TN_k, TP_k, $t.n.p._k$, and $t.n._k$ if all its minors up to order k are nonnegative, positive, nonpositive, and negative, respectively.*

The following interval property can be deduced from Theorem 1 in [15]* on the interval property of TP matrices using a result by Karlin [27, Chapter 2, Theorem 3.3] which was first proved by Fekete [17] in 1912 for TP matrices, and subsequently extended by Schoenberg [40, Lemma 1] in 1955 to TP_k matrices, see also Corollary 3.1.6 in [12]*. For a different proof of the interval property see [11, Theorem B] which employs a sign non-reversal property.

IP 2.6. *The set of the TP_k $m \times n$ matrices has the interval property with respect to the two corner matrices.*

IP 2.7. *[9, Theorem B] The set of the $t.n._k$ $m \times n$ matrices has the interval property with respect to the two corner matrices.*

The interval property of the TN_k and the $t.n.p._k$ matrices requires in general considerably more vertex matrices, see IP 4.9 and 4.10. If $m = n = k$, we obtain the (square) totally nonnegative matrices (denoted by TN). IP 3.2.3(a)* tells that the nonsingular TN matrices possess the interval property with respect to the two corner matrices. A weakening of the nonsingularity assumption was presented in Theorem 3.2*. Recently, a different weakening of the nonsingularity assumption was given in [1]. Besides the property of being TN and a further property like the descending rank conditions [24], it is required that certain rows and columns are linear independent.

Definition 2.7. *A matrix $A \in \mathbb{R}^{n,n}$ is an $NsTN^-$-matrix if it is TN_{n-1} and has a negative determinant.*

This set is closely related to the nonsingular totally nonpositive matrices which also has the interval property, see IP 3.2.5*: The matrix $A \in \mathbb{R}^{n,n}$ is $NsTN^-$ if and only if $JA^{-1}J$ is totally nonpositive [23, p. 1247], where $J := \mathrm{diag}(1, -1, 1, \ldots, (-1)^{n-1})$. For examples of $NsTN^-$ matrices related to Cauchy-Vandermonde matrices, see [42].

IP 2.8. *[2, Theorem 5.2] The set of the $NsTN^-$-matrices has the interval property with respect to the two corner matrices.*

From IP 2.8, one can deduce the interval property of further classes of nonsingular sign regular ($NsSR$) matrices, cf. IP 3.2.6*.

IP 2.9. *[2, Theorem 5.3] The $n \times n$ $NsSR$ matrices with one of the following signatures $\epsilon = (\epsilon_i)_{i=1}^n$ have the interval property:*

(i) $\epsilon_i = (-1)^i$, $i = 1, \ldots, n-1$, $\epsilon_n = (-1)^{n-1}$,

(ii) $\epsilon_i = (-1)^{\frac{i(i-1)}{2}}$, $i = 1, \ldots, n-1$, $\epsilon_n = (-1)^{\frac{n(n-1)}{2}+1}$,

(iii) $\epsilon_i = (-1)^{\frac{i(i+1)}{2}}$, $i = 1, \ldots, n-1$, $\epsilon_n = (-1)^{\frac{n(n+1)}{2}+1}$.

3 Matrix Properties Which Require $O(n)$ Vertex Matrices

In this section we consider matrix intervals with respect to the usual entry-wise partial order.

We start with a matrix class which is also related to the linear complementarity problem, see, e.g., [36]. We need the following notation. For a matrix $A = [a_{ij}] \in \mathbb{R}^{n,n}$ define for $i = 1, \ldots, n$, $r_i^+ := \max\{0, a_{ij} | i \neq j\}$.

Definition 3.1. *[37] A matrix $A = (a_{ij}) \in \mathbb{R}^{n,n}$ is called B-*matrix *if for $i = 1, \ldots, n$*

(a) $\sum_{j=1}^{n} a_{ij} > 0$,

(b) for all $k \in \{1, \ldots, n\} \setminus \{i\} : \frac{1}{n} \sum_{j=1}^{n} a_{ij} \geq a_{ik}$.

IP 3.1. *[30, Proposition 3.12 and 3.13] The set of the B-matrices has the interval property with respect to the set of the vertex matrices $A^{(m)} = (a_{ij}^{(m)})$, $m = 1, \ldots, n$, defined by*

$$a_{ij}^{(m)} := \begin{cases} \overline{a}_{ij} & \textit{if } i \neq m, j = m, \\ \underline{a}_{ij} & \textit{otherwise.} \end{cases}$$

This set of n vertex matrices is minimal, i.e., it cannot be replaced by a proper nonempty subset.

The set of the B-matrices is contained in the set of the doubly B-matrices which are defined as follows.

Definition 3.2. *[38] A matrix $A = (a_{ij}) \in \mathbb{R}^{n,n}$ is called a* doubly B-matrix *if for $i = 1, \ldots, n$,*

(a) $a_{ii} > r_i^+$,

(b) for all $j \in \{1, \ldots, n\} \setminus \{i\} : (a_{ii} - r_i^+)(a_{jj} - r_j^+) > (\sum_{k\neq i}(r_i^+ - a_{ii}))(\sum_{k\neq j}(r_j^+ - a_{ik}))$.

The interval property of doubly B-matrices can be found for $n \geq 4$ [1] in [30, Propositions 4.10 and 4.12] which requires $\frac{n}{2}(n-1)^3$ vertex matrices (which are furthermore minimal) and Proposition 4.13 which requires n^3 vertex matrices. A further generalization of the B-matrices are the B_π^R - *matrices*, which have been introduced in [35]. These matrices are P-matrices. An interval property for these matrices can be found in [30, Proposition 5.9].

In the remainder of this subsection we consider only symmetric matrices and consequently only those matrices in a matrix interval which are symmetric, see Subsection 4.3(b)*.

An important subclass of the inverse M-matrices are the strictly ultrametric matrices, see [15], [25, Section 5.28], for which the number of the needed vertex matrices can be significantly reduced.

Definition 3.3. *A symmetric nonnegative matrix $A = (a_{ij}) \in \mathbb{R}^{n,n}$ is said to be* ultrametric *if the following two conditions hold:*

(a) $a_{ik} \geq \min\{a_{ij}, a_{jk}\}$, *for all* $i, j, k \in \{1, \ldots, n\}$;

(b) $a_{ii} \geq \max_{k\neq i} a_{ik}$, *for all* $i = 1, \ldots, n$.

If the inequality in (b) is strict, A is called strictly ultrametric. *If in this case $n = 1$, (b) is replaced by $a_{11} > 0$.*

Let $[A] = [\underline{A}, \overline{A}]$ be a matrix interval in $\mathbb{R}^{n,n}$ with $A \geq 0$ and let $\underline{A}$ and $\overline{A}$ be symmetric. We are making again use of the representation of $[A]$ as an interval matrix, i.e., $[A] = [\underline{a}_{ij}, \overline{a}_{ij}]_{ij=1}^{n}$, see (2)*. Define the n element matrices $A^{(m)} = (a_{ij}^{(m)})$ of $[A]$, $m = 0, \ldots, n-1$, as follows: Put all entries $a_{ij}^{(m)} := \overline{a}_{ij}$ with the exception of the entries $a_{i,i+m}^{(m)}$ which are defined as $a_{i,i+m}^{(m)} := \underline{a}_{i,i+m}$, $i = 1, \ldots, n - m$. If the matrices $A^{(m)}$,

[1] For n=2,3, see Remark 4.11 therein.

$m = 0, \ldots, n-1$, fulfill the conditions (a) and (b), then all symmetric matrices in $[A]$ are (strictly) ultrametric. However, the matrices $A^{(m)}$ are in general not symmetric. To obtain a valid interval property, at the expense of $\lfloor \frac{n-1}{2} \rfloor$ more vertex matrices, some technical expenditure is needed to avoid the encounter of the lower endpoints of two component intervals in a common row or column (regardless whether the intervals are in the upper or the lower triangular part). Keep the matrix $A^{(0)}$. Again we proceed diagonal by diagonal from the left to the right and define only the upper triangular part (including the main diagonal) and complete the lower triangular part by the requirement that the vertex matrices are symmetric.

Define the matrices $B^{(m)} = (b_{ij}^{(m)})$, $C^{(m)} = (c_{ij}^{(m)})$, $m = 1, \ldots, \lfloor \frac{n-1}{2} \rfloor$, as follows: Put

$$b_{ij}^{(m)} = c_{ij}^{(m)} := \overline{a}_{ij}, \text{ if } j \neq i+m.$$

If m is odd, put

for $k = 1, 2, \ldots, \lceil \frac{n-m}{2} \rceil$ with $i = 2k-1$, $b_{i,i+m}^{(m)} := \underline{a}_{i,i+m}$, $c_{i,i+m}^{(m)} := \overline{a}_{i,i+m}$;

for $k = 1, 2, \ldots, \lfloor \frac{n-m}{2} \rfloor$ with $i = 2k$, $b_{i,i+m}^{(m)} := \overline{a}_{i,i+m}$, $c_{i,i+m}^{(m)} := \underline{a}_{i,i+m}$.

If m is even and not a multiple of 4, put for $p = 1, 2, \ldots$,

$$b_{i,i+m}^{(m)} := \begin{cases} \underline{a}_{i,i+m}, & \text{if } i = 4p-3, 4p, \\ \overline{a}_{i,i+m}, & \text{if } i = 4p-2, 4p-1. \end{cases}$$

If m is even and a multiple of 4, put for $p = 0, 1, 2, \ldots$, $l = 1, 2, \ldots, m$,

$$b_{i,i+m}^{(m)} := \begin{cases} \underline{a}_{i,i+m}, & \text{if } i = 2pm + l, \\ \overline{a}_{i,i+m}, & \text{if } i = (2p+1)m + l. \end{cases}$$

Define the matrices $D^{(m)} = (d_{ij}^{(m)})$, $m = \lfloor \frac{n-1}{2} \rfloor + 1, \ldots, n-1$, as follows

$$d_{ij}^{(m)} := \overline{a}_{ij}, \text{ if } j \neq i+m,$$

$$d_{i,i+m}^{(m)} := \underline{a}_{i,i+m}, \text{ if } i = 1, \ldots, n-m.$$

IP 3.2. *The set of the $n \times n$ (strictly) ultrametric matrices has the interval property with respect to the set of vertex matrices*

$$\left\{ A^{(0)}; B^{(m)}, C^{(m)}, m = 1, \ldots, \lfloor \frac{n-1}{2} \rfloor; D^{(m)}, m = \lfloor \frac{n-1}{2} \rfloor + 1, \ldots, n-1 \right\}.$$

Proof. Let $[A]$ be a matrix interval in $\mathbb{R}^{n,n}$, and assume that the vertex matrices $A^{(0)}, B^{(m)}, C^{(m)}$, $m = 1, \ldots, n-1$, are (strictly) ultrametric. Let A be any symmetric matrix in $[A]$. Since all the matrices under consideration are symmetric, it suffices to check conditions (a) and (b) only for the entries in the upper triangular part including the main diagonal of A. The assumption that $A^{(0)}$ is ultrametric implies that for all $i = 1, \ldots, n$

$$a_{ii} \geq \underline{a}_{ii} \geq \min_{j \neq i} \{\overline{a}_{ij}, \overline{a}_{ji}\} \geq \min_{j \neq i} \{a_{ij}\};$$

$$a_{ii} \geq \underline{a}_{ii} \geq (>) \max_{k \neq i} \overline{a}_{ik} \geq \max_{k \neq i} a_{ik}.$$

Therefore, conditions (a) and (b) are fulfilled for the diagonal entries of A. By similar inequalities obtained by using the other vertex matrices, it follows that condition (a) is also satisfied for the off-diagonal entries of A. □

4 Matrix Properties Which Require at Most About 2^{n-1} or at Most 2^{2n-1} Vertex Matrices

First we consider matrix intervals with respect to the usual entry-wise partial order. For the reader's convenience, we recall from [18] the definition of two special sets of vertex matrices.

Each matrix interval $[A] = [\underline{A}, \overline{A}]$ can be represented as $\{A \in \mathbb{R}^{n,n} \mid | A - A_c | \leq \Delta\}$, where $A_c = \frac{1}{2}(\overline{A} + \underline{A})$ is the *midpoint matrix* and $\Delta = \frac{1}{2}(\overline{A} - \underline{A})$ is the *radius matrix*, in particular, $\underline{A} = A_c - \Delta$ and $\overline{A} = A_c + \Delta$.

With $Y_n = \{y \in \mathbb{R}^n \mid | y_i |= 1,\ i = 1, \ldots, n\}$ and $T_y = \operatorname{diag}(y_1, y_2, \ldots, y_n)$ we define the set $V_1([A])$ of matrices $A_{yz} = A_c - T_y \Delta T_z$ for all $y, z \in Y_n$. The definition implies that for all $i, j = 1, \ldots, n$,

$$(A_{yz})_{ij} = (A_c)_{ij} - y_i(\Delta)_{ij} z_j = \begin{cases} \overline{a}_{ij} & \text{if} \quad y_i z_j = -1, \\ \underline{a}_{ij} & \text{if} \quad y_i z_j = 1, \end{cases} \tag{4.1}$$

so that all matrices A_{yz} are vertex matrices. The cardinality of $V_1([A])$ is at most 2^{2n-1}. In the following subsection, we consider the set $V_2([A])$ which is obtained from $V_1([A])$ if we choose $y = z$. The cardinality is then reduced to at most 2^{n-1}.

4.1 Properties Requiring at Most About 2^{n-1} Vertex Matrices

We continue with a matrix class which is related to the linear complementarity problem, see [14, Section 3.5].

Definition 4.1. *A matrix $A \in \mathbb{R}^{n,n}$ is called* column sufficient *if for all vectors $x \in \mathbb{R}^n$, $x_i(Ax)_i \leq 0$, $i = 1, \ldots, n$, implies $x_i(Ax)_i = 0$, $i = 1, \ldots, n$.*

IP 4.1. *[20, Proposition 11][2] The set of the column sufficient matrices has the interval property with respect to the set V_2 of vertex matrices.*

A matrix A is called *row sufficient* if A^T is column sufficient, and if A is both column and row sufficient it is termed *sufficient*. Obviously, IP 4.1 carries over to the set of (row) sufficient matrices.

Also the next classes are related to the linear complementarity problem, see, e.g., [34].

Definition 4.2. *A matrix $A \in \mathbb{R}^{n,n}$ is an N*-matrix *if all its principal minors are negative, it is an N*-matrix of the first category *if it has at least one positive entry. Otherwise, A is of the* second category. *Let α be a nonempty proper subset of $\{1, \ldots, n\}$ and $\alpha^c := \{1, \ldots, n\} \setminus \alpha$, where we consider the sequence associated with α^c as increasingly ordered. The matrix A is an N*-matrix of the first category with respect to α *if*

$$A[\alpha],\ A[\alpha^c] < 0, \ \ and\ \ A[\alpha|\alpha^c],\ A[\alpha^c|\alpha] > 0. \tag{4.2}$$

[2] Here a characterization of the column sufficient matrices is required, see [14, Proposition 3.5.9].

In fact, if A is an N-matrix of the first category, then there is a permutation matrix P such that PAP^T is of the form (4.2), see [34, Theorem 4.3].

We define $e^\alpha \in \mathbb{R}^n$ as the vector with

$$(e^\alpha)_i := 1, \text{ if } i \in \alpha, \textit{ and } (e^\alpha)_i := -1, \text{ otherwise}, \; i = 1, \dots, n, \tag{4.3}$$

and put $e := e^{\{1,\dots,n\}}$, i.e., e is the vector the components of which are all 1.

Theorem 4.1. *[10, Theorem 3.6] Let $[A] = [\underline{A}, \overline{A}]$ be a matrix interval in $\mathbb{R}^{n,n}$ with $\overline{a}_{ii} < 0$, $i = 1, \dots, n$. Then $[A]$ is an N-matrix of the second category if and only if the vertex matrices in $V_2 \setminus \{\underline{A}\}$ are N-matrices of the second category.*

Theorem 4.2. *[10, Theorem 3.7] Let α be a nonempty proper subset of $\{1, \dots, n\}$, and let $[A] = [\underline{A}, \overline{A}]$ be a matrix interval in $\mathbb{R}^{n,n}$ with $\overline{a}_{ii} < 0$, $i = 1, \dots, n$. Then $[A]$ is an N-matrix of the first category with respect to α if and only if the vertex matrices in $V_2([A])$ generated by $z \in Y^n \setminus \{\pm e^\alpha\}$ are N-matrices of the first category with respect to α.*

We consider now two subsets of the almost P-matrices, see Definition 2.2. The following classification into two categories is motivated by the fact that a matrix (of least order 2) is an almost P-matrix if and only if its inverse matrix is an N-matrix, see [29, Lemma 2.4].

Definition 4.3. *Let α be a nonempty proper subset of $\{1, \dots, n\}$. An almost P-matrix is an* almost P-matrix of the first category with respect to α *if A^{-1} is an N-matrix of the first category with respect to α. It is an* almost P-matrix of the second category *if A^{-1} is an N-matrix of the second category.*

We put $M_\alpha := A_c + T_{e^\alpha} \triangle T_{e^\alpha}$. For the definition of e^α see (4.3).

IP 4.2. *[10, Theorem 4.3] The set of the $n \times n$ almost P-matrices of the second category has the interval property with respect to the set V_2 of vertex matrices together with the upper corner matrix.*

IP 4.3. *[10, Theorem 4.5] The set of the $n \times n$ almost P-matrices of the first category with respect to a fixed nonempty proper subset α of $\{1, \dots, n\}$ has the interval property with respect to the set $V_2 \cup \{M_\alpha\}$.*

We consider now the set of the inverse M-matrices, i.e., regular matrices having an M-matrix as inverse matrix (Definition 4.1*). For a comprehensive survey on this class see [25, Chapter 5]. By IP 4.1* and the remark following it, this set has the interval property with respect to a set of vertex matrices of cardinality at most 2^{n^2-n}. It has been conjectured in [19, Conjecture 1] that the cardinality of the needed set of vertex matrices can be reduced to $2n^2$. The following interval property reveals that it can be reduced to 2^{n-1} but it leaves the conjecture still open.

IP 4.4. *The set of the inverse M-matrices has the interval property with respect to the set V_2 of vertex matrices.*

Proof. We use the following characterization of nonnegative matrices having a special property called the *Minkowski property* [41, Theorem 3], see also [28, Chapter 3, Theorem 6]: The inverse matrix of $A \in \mathbb{R}^{n,n}$ has nonnegative diagonal entries and nonpositive off-diagonal entries if and only if for any nonempty proper subset α of $\{1, \ldots, n\}$ of cardinality r and any given positive vector $y \in \mathbb{R}^{n-r}$, the following system has a positive solution $x \in \mathbb{R}^r$,

$$(T_{e^\alpha} A T_{e^\alpha}) w > 0, \tag{4.4}$$

where the vector e^α is introduced in (4.3) and the components of $w \in \mathbb{R}^n$ are composed of the r components of x and the $n-r$ components of y. Now let $[A]$ be a matrix interval in $\mathbb{R}^{n,n}$ and assume that all its vertex matrices from V_2 are inverse M-matrices. Choose any $A \in [A]$ and consider the system (4.4) with any given positive vector $y \in \mathbb{R}^{n-r}$. Since A is nonnegative, we obtain for any positive vector $x \in \mathbb{R}^r$,
if $i \in \alpha$:

$$\sum_{j \in \alpha} a_{ij} x_j - \sum_{j \notin \alpha} a_{ij} y_j \geq \sum_{j \in \alpha} \underline{a}_{ij} x_j - \sum_{j \notin \alpha} \overline{a}_{ij} y_j, \tag{4.5a}$$

if $i \notin \alpha$:

$$-\sum_{j \in \alpha} a_{ij} x_j + \sum_{j \notin \alpha} a_{ij} y_j \geq -\sum_{j \in \alpha} \overline{a}_{ij} x_j + \sum_{j \notin \alpha} \underline{a}_{ij} y_j. \tag{4.5b}$$

We may conclude that we have found a vertex matrix in V_2 such that the system of the form (4.4) associated with this vertex matrix has a positive solution for the given positive vector y. By (4.5), the system (4.4) has the same positive solution. Therefore, the nonnegative matrix A has the Minkowski property. By [28, Chapter 3, Lemma 10], each nonnegative matrix possessing the Minkowski property as well as its inverse matrix are P-matrices. Since A^{-1} is a P-matrix having nonpositive off-diagonal entries, A is an inverse M-matrix. □

For a matrix interval $[A]$ in $\mathbb{R}^{n,n}$, we define the set $V_3([A])$ of its vertex matrices as the set of matrices A_{zz},

$$A_{zz} := A_c + T_z \triangle T_z, \text{ for all } z \in Y_n;$$

this means that for $y = -z$ in the explicit representation (4.1) of the matrices contained in $V_2([A])$, the role of the lower and upper endpoints of the coefficient intervals has to be interchanged. The next two matrix classes play also a role in mathematical economics.

Definition 4.4. *A matrix $A \in \mathbb{R}^{n,n}$, $n \geq 2$, is called* semi-PN-matrix *if every principal minor of order r of A has sign $(-1)^{r-1}$, $r = 2, \ldots, n$.*

By a similar proof as for IP 4.4 and using [41, Theorem 2], see also [28, Chapter 3, Theorem 5], one shows the following interval property which has been already presented in [11*, p. 38].

IP 4.5. *The set of the nonnegative semi-PN-matrices has the interval property with respect to the set of vertex matrices in V_3 which are different from the right corner matrix.*

Definition 4.5. *A square matrix A has the* Metzler property*, if its inverse matrix has nonpositive diagonal entries and nonnegative off-diagonal entries.*

By a similar proof as for IP 4.4 and using [41, Theorem 4], see also [28, Chapter 3, Theorem 7], one shows the following interval property.

IP 4.6. *The set of the nonnegative matrices with positive off-diagonal entries which possess the Metzler property has the interval property with respect to the set V_3 of vertex matrices.*

Properties and applications in optimization and statistics of the next matrix class can be found in [4, Chapter 4].

Definition 4.6. *A symmetric matrix $A \in \mathbb{R}^{n,n}$ is said to be* conditionally positive (negative) semidefinite *if $x^T Ax \geq 0$ $(\leq 0,)$ for all vectors $x \in \mathbb{R}^n$ with $x^T e = 0$, A is called* conditionally positive (negative) definite *if the inequality is strict for all vectors $x \in \mathbb{R}^n$ with $x \neq 0$ and $x^T e = 0$.*

IP 4.7. *The set of the conditionally positive (negative) semidefinite matrices has the interval property with respect to the set V_2 (V_3) of vertex matrices.*

Proof. We apply the following fact [22, p. 144]: A symmetric matrix $B = (b_{ij}) \in \mathbb{R}^{n,n}$ is conditionally positive semidefinite if and only if the *Hadamard exponential* $e^{\circ tB} := (e^{tb_{ij}})$ is positive semidefinite for all $t \geq 0$. We first prove the statement for conditionally positive semidefinite matrices. Let $[A] = [\underline{A}, \overline{A}]$ be a matrix interval in $\mathbb{R}^{n,n}$ and assume that both corner matrices are symmetric. Furthermore, assume that all vertex matrices in V_2 are conditionally positive semidefinite. Let $A \in [A]$ be symmetric. Then it follows that

$$e^{\circ t\underline{A}} \leq e^{\circ tA} \leq e^{\circ t\overline{A}}, \text{ for all } 0 \leq t.$$

There is a one-to-one correspondence between the matrices in $V_2([A])$ and the matrices in $V_2([e^{\circ t\underline{A}}, e^{\circ t\overline{A}}])$, where the later vertex matrices are positive semidefinite for all $0 \leq t$. By IP 4.5*, $e^{\circ tA}$ is positive semidefinite for all $0 \leq t$, and we can conclude that A is conditionally positive semidefinite. The statement for conditionally negative semidefinite matrices follows now by using $[-\overline{A}, -\underline{A}]$ instead of $[A]$ and the set V_3 instead of V_2. □

The question whether an analogous interval property is true for the conditionally positive definite matrices is open. Unfortunately, the necessary and sufficient condition for a matrix to be conditionally positive semidefinite used in the proof of IP 4.7 does not carry over to the conditionally positive definite matrices: If a matrix A is conditionally positive definite then, in fact, its Hadamard exponential $e^{\circ A}$ is positive definite, see [39, Corollary 2.6], but the reverse statement is not true. A counterexample is provided by the matrix

$$A := \begin{pmatrix} 6 & 2 & 1 \\ 2 & 1 & 4 \\ 1 & 4 & 12 \end{pmatrix},$$

which is not conditionally positive definite while $e^{\circ A}$ is positive definite.

Definition 4.7. *A symmetric, positive semidefinite, and nonnegative matrix $A = (a_{ij})$ is called* infinitely divisible *if its* Hadamard power $A^r := (a_{ij}^r)$ *is positive semidefinite for each nonnegative number r.*

For properties and many examples of this matrix class the reader is referred to [5].

IP 4.8. *The set of the positive and infinitely divisible matrices has the interval property with respect to the set V_2 of vertex matrices.*

Proof. The proof is similar to the one of IP 4.7. Here we use the following fact [21, Corollary 1.6]: Let the matrix $B = (b_{ij}) \in \mathbb{R}^{n,n}$ be entry-wise nonzero. Then B is infinitely divisible if and only if B is symmetric and positive, and its *Hadamard logarithm* $\log^\circ(B) := (\log(b_{ij}))$ is conditionally positive semidefinite. Let $[A] = [\underline{A}, \overline{A}]$ be a matrix interval in $\mathbb{R}^{n,n}$ and assume that $\underline{A} > 0$ and both corner matrices are symmetric. Furthermore, assume that all vertex matrices in V_2 are infinitely divisible. Let $A \in [A]$ be symmetric. Then it follows that

$$\log^\circ(\underline{A}) \leq \log^\circ(A) \leq \log^\circ(\overline{A}).$$

We conclude by IP 4.7 that $\log^\circ(A)$ is conditionally positive semidefinite, and therefore, A is infinitely divisible. □

4.2 Properties Requiring at Most 2^{2n-1} Vertex Matrices

IP 4.9. *[11, Theorem D]* [3] *The set of the TN_k matrices has the interval property with respect to the set V_1 of vertex matrices.*

IP 4.10. *[9, Theorem 5.3]* [3] *The set of the t.n.p.k matrices has the interval property with respect to the set V_1 of vertex matrices.*

Now we consider matrix intervals in $\mathbb{R}^{n,n}$ with respect to the checkerboard partial order. The example in [15]* and Example 3.2.5 in [12]* document that the set of the $TN = TN_n$ matrices does not have the interval property with respect to the two corner matrices. The situation is different for tridiagonal matrices.

IP 4.11. *The set of the tridiagonal TN_k matrices has the interval property with respect to the two corner matrices.*

Proof. It follows from p. 6 in [12]* that a tridiagonal matrix is TN_k if all its off-diagonal entries and all its contiguous principal minors up to order k are nonnegative. Let $[A]$ be a matrix interval in $\mathbb{R}^{n,n}$. We are making again use of the representation of $[A]$ as an interval matrix, i.e., $[A] = [\underline{a}_{ij}, \overline{a}_{ij}]_{ij=1}^n$. Assume that the two corner matrices are tridiagonal and TN_k, and let $A \in [A]$. Since the upper corner matrix is TN_k, it follows that

$$a_{i,i+1} \geq \underline{a}_{i,i+1} \geq 0, \;\; a_{i+1,i} \geq \underline{a}_{i+1,i} \geq 0, \;\; i = 1, \ldots, n-1.$$

Since each contiguous principal submatrix of A is in turn a tridiagonal matrix, we may apply IP 3.2.4* to conclude that each such submatrix up to order k is TN, too. □

Acknowledgments

We thank Projesh Nath Choudhury, Milan Hladík, and Apoorva Khare for their careful reading of the manuscript and their valuable comments.

---

[3] As proven therein, the statement is also true for $m \times n$ matrices with $k \leq m, n$.

References

[1] M. Adm, K. Al Muhtaseb, A. Abedel Ghani, and J. Garloff. Relaxing the nonsingularity assumption for intervals of totally nonnegative matrices. *Electron. J. Linear Algebra*, 36:106–123, 2020.

[2] M. Adm and J. Garloff. Characterization, perturbation, and interval property of certain sign regular matrices. *Linear Algebra Appl.*, 612:146-161, 2021.

[3] G. Alefeld and U. Schäfer. Iterative methods for linear complementarity problems with interval data. *Computing*, 70:235-259, 2003.

[4] R.B. Bapat and T.E.S. Raghavan. *Nonnegative Matrices and Applications.* Encyclopedia Math. Appl. Vol. 64, Cambridge Univ. Press, Cambridge, UK, 1997.

[5] R. Bhatia. Infinitely divisible matrices. *Amer. Math.*, Monthly 113(3):221-235, 2006.

[6] M. Černý. Interval matrices with Monge property. *Appl. Math.*, 65(5):619-643, 2020.

[7] M. Chatterjee and K.C. Sivakumar. Intervals of H-matrices and inverse M-matrices. *Linear Algebra Appl.*, 614:24-43, 2021.

[8] P.N. Choudhury. Characterizing total positivity: single vector tests via linear complementarity, sign non-reversal, and variation diminution. arxiv: 2103.05624.

[9] P.N. Choudhury. Characterizing total negativity and testing their interval hulls. arxiv: 2103.13384.

[10] P.N. Choudhury, M.R. Kannan. Interval hulls of N-matrices and almost P-matrices. *Linear Algebra Appl.*, 617:27-38, 2021.

[11] P.N. Choudhury, M.R. Kannan, and A. Khare. Sign non-reversal property for totally positive matrices and testing total positivity of their interval hull. *Bull. Lond. Math. Soc.*, to appear.

[12] P.N. Choudhury, M.R. Kannan, and K.C. Sivakumar. New contributions to semipositive and minimally semipositive matrices. *Electron. J. Linear Algebra*, 34:35-53, 2018.

[13] R.W. Cottle. A field guide to the matrix classes found in the literature of the linear complementarity problem. *J. Global Optim.*, 46(4):571-580, 2010.

[14] R.W. Cottle, J.S. Pang, and R.E. Stone. *The Linear Complementarity Problem.* Classics Appl. Math. Vol. 60, SIAM, Philadelphia, 2009.

[15] C. Dellacherie, S. Martinez, J. San Martin. *Inverse M-Matrices and Ultrametric Matrices.* Lect. Notes in Math. Vol. 2118. Springer, Cham, 2014.

[16] K. Fan. Some matrix inequalities. *Abh. Math. Semin. Univ. Hambg.*, 29:185-196, 1966.

[17] M. Fekete and G. Pólya. Über ein Problem von Laguerre. *Rend. Circ. Mat. Palermo*, 34:89-120, 1912.

[18] J. Garloff, M. Adm, and J. Titi. A survey of classes of matrices possessing the interval property and related properties. *Reliab. Comput.*, 22:1-14, 2016.

[19] M. Hladík. An overview of polynomially computable characteristics of special interval matrices. In: O. Kosheleva, S.P. Shary, G. Xiang, and R. Zapatrin. *Beyond Traditional Probabilistic Data Processing Techniques: Interval, Fuzzy etc. Methods and Their Applications*. Springer Stud. Comput. Intell. Vol. 835, pp. 295-310, Springer, Cham, 2020.

[20] M. Hladík. Stability of the linear complementarity problem properties under interval uncertainty. *CEJOR Cent. Eur. J. Oper. Res.*, to appear, DOI 10.1007/s10100-021-00745-6.

[21] R.A. Horn. The theory of infinitely divisible matrices and kernels. *Trans. Amer. Math. Soc.*, 136:269-286, 1969.

[22] R.A. Horn. The Hadamard product. *Proc. Sympos. Appl. Math.*, 40:87-169, 1990.

[23] R. Huang. A test and bidiagonal factorization for certain sign regular matrices. *Linear Algebra Appl.*, 438:1240-1251, 2013.

[24] C.R. Johnson, D.D.Olesky, and P. van den Driessche. Successively ordered elementary bidiagonal factorization. *SIAM J. Matrix Anal. Appl.*, 22(4):1079-1088, 2001.

[25] C.R. Johnson, R.L. Smith, and M.J. Tsatsomeros. *Matrix Positivity*. Cambridge Tracts in Math. Vol. 221, Cambridge Univ. Press, Cambridge, UK, 2020.

[26] G.A. Johnson. A generalization of N-matrices. *Linear Algebra Appl.*, 48:201-217, 1982.

[27] S. Karlin. *Total Positivity*, Vol. I. Stanford University Press, Stanford, CA, 1968.

[28] M.C. Kemp and Y. Kimura. *Introduction to Mathematical Economics*. Springer, New York, Heidelberg, Berlin, 1978.

[29] M. Kojima and R. Saigal. On the number of solutions to a class of linear complementarity problems. *Math. Program*, 17:136-139, 1979.

[30] M. Lorenc. Special classes of P-matrices in the interval setting. Bachelor thesis, Department of Applied Mathematics, Charles University, Prague, 2021.

[31] H.Q. Ma, J.P. Xu, and N.J. Huang. An iterative method for a system of linear complementarity problems with perturbations and interval data. *Appl. Math. Comput.*, 215:175-184, 2009.

[32] G. Mayer. *Interval Analysis*. De Gruyter Stud. Math. Vol. 65, De Gruyter, Berlin, Boston, 2017.

[33] J. Miao. Ky Fan's N-matrices and linear complementarity problems. *Math. Program*, 61:351-356, 1993.

[34] S.R. Mohan and R. Sridhar. On characterizing N-matrices using linear complementarity. *Linear Algebra Appl.*, 160:231-245, 1992.

[35] M. Neumann, J.M. Peña, and O. Pryporova. Some classes of nonsingular matrices and applications. *Linear Algebra Appl.*, 438(4):1936-1945, 2013.

[36] H. Orera and J.M. Peña. Error bounds for linear complementarity problems of B_π^R*-matrices*. *Comput. Appl. Math.*, 40(3): article 94, 2021.

[37] J.M. Peña. A class of P-matrices with applications to the localization of the eigenvalues of a real matrix. *SIAM J. Matrix Anal. Appl.*, 22(4):1027-1037, 2001.

[38] J.M. Peña. On an alternative to Gerschgorin circles and ovals of Cassini. *Numer. Math.*, 95(2): 337-345, 2003.

[39] R. Reams. Hadamard inverses, square roots and products of almost semidefinite matrices. *Linear Algebra Appl.*, 288:35-43, 1999.

[40] I.J. Schoenberg. On the zeros of the generating functions of multiply positive sequences and functions. *Ann. of Math. (2)*, 62(3):447-471, 1955.

[41] Y. Uekawa, M.C. Kemp, and L.L. Wegge. P- and PN-matrices, Minkowski- and Metzler-matrices, and generalizations of the Stolper-Samuelson and Samuelson-Rybczynski theorems. *J. Int. Econ.*, 3:53-76, 1972.

[42] Z. Yang. Accurate computations of eigenvalues of quasi-Cauchy-Vandermonde matrices. *Linear Algebra Appl.*, to appear.