Computing the United Solution Set to an Interval Linear System Is NP-Hard – Even When All Coefficients Are Known With the Same Accuracy*

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Abstract

When the coefficients of a linear system are known with interval uncertainty, instead of a *single* solution, we have the whole *set* of possible solutions – known as the *united solution set*. It is known that in general, computing this united solution set is NP-hard. There exist several proofs of this NP-hardness; all known proofs use examples with intervals of different width – corresponding to different accuracy in measuring different coefficients. We show that the problem remains NP-hard even if we limit ourselves to situations when all the coefficients are known with the same accuracy.

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1 Introduction

Solving a linear system under interval uncertainty: formulation of the problem. In practice, we are often interested in the values of the quantities $y_1, \ldots, y_m$ which are difficult (or impossible) to measure directly. In many cases, to find the values y_j, we can use the fact that these values satisfy a system of linear equations $\sum\limits_{j=1}^{m} a_{i,j} \cdot y_j = b_i$, $1 \le i \le p$, in which each of the coefficients $a_{i,j}$ and b_i is either known exactly, or can be (directly) measured.

When all the coefficients $a_{i,j}$ and b_i are known exactly, and when the matrix $a_{i,j}$ is regular, then we can solve this linear system and find the exact values of the unknowns y_j. In practice, some of coefficients come from measurements, and measurements are never absolutely accurate, there is always a measurement error.

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Because of the measurement error, for each measured quantity q, the measurement result $\widetilde{q}$ is, in general, different from the actual (unknown) value q. Usually, we know the upper bound Δ on the measurement error $\Delta q \stackrel{\text{def}}{=} \widetilde{q} - q$: $|\Delta q| \le \Delta$. In this situation, the only information that we have about the actual value q is that this value belongs to the interval $\boldsymbol{q} = [\underline{q}, \overline{q}] \stackrel{\text{def}}{=} [\widetilde{q} - \Delta, \widetilde{q} + \Delta]$; see, e.g., [6].

A measurement usually results in a rational value $\widetilde{r}$, i.e., the ratio of two integers n_1/n_2. Moreover, it is usually a binary number, so it is binary-rational – i.e., of the type $n_1/2^{n_2}$. The bound Δ is also usually rational – binary or decimal. Thus, the bounds of the resulting interval $[\underline{q}, \overline{q}] = [\widetilde{q} - \Delta, \widetilde{q} + \Delta]$ are rational.

So, based on the measurements, we only know the intervals $\boldsymbol{a}_{i,j}$ and $\boldsymbol{b}_i$ of possible values of the corresponding coefficients. For different values $a_{i,j} \in \boldsymbol{a}_{i,j}$ and $b_i \in \boldsymbol{b}_i$, we have, in general, different solutions y_j.

It is therefore desirable to find, for each j, the interval of possible values y_j. Thus, we arrive at the following problem.

Definition 1. *By an* interval linear system*, we mean a tuple consisting of integers m and p and intervals $\boldsymbol{a}_{i,j}$ and $\boldsymbol{b}_i$, $1 \le i \le p$, $1 \le j \le m$ with rational bounds. A system will also be denoted by $\sum\limits_{j=1}^{m} \boldsymbol{a}_{i,j} \cdot y_j = \boldsymbol{b}_i$.*

Definition 2. *We say that a tuple $y = (y_1, \ldots, y_m)$ is a* possible solution *to the interval linear system if for some $a_{i,j} \in \boldsymbol{a}_{i,j}$ and $b_i \in \boldsymbol{b}_i$, we have $\sum\limits_{j=1}^{m} a_{i,j} \cdot y_j = b_i$ for all i.*

Comment. The term "possible solution" comes from [3]. In [1], such solutions are called *weak solutions*.

Definition 4. *The set of all possible solutions is called a* united solution set.

Definition 5. *Let $\varepsilon > 0$ be a real number.*

- *We say that a number $\widetilde{v}$ is an* ε-approximation *to a number v if $|\widetilde{v} - v| \le \varepsilon$.*
- *We say that an interval $\left[\widetilde{\underline{v}}, \widetilde{\overline{v}}\right]$ is an* ε-approximation *to an interval $[\underline{v}, \overline{v}]$ if $|\widetilde{\underline{v}} - \underline{v}| \le \varepsilon$ and $\left|\widetilde{\overline{v}} - \overline{v}\right| \le \varepsilon$.*
- *We say that a set $\left[\widetilde{\underline{v}}_1, \widetilde{\overline{v}}_1\right] \times \ldots \times \left[\widetilde{\underline{v}}_m, \widetilde{\overline{v}}_m\right]$ is an* ε-approximation *to the set $[\underline{v}_1, \overline{v}_1] \times \ldots \times [\underline{v}_m, \overline{v}_m]$ if for ever i, the interval $\left[\widetilde{\underline{v}}_i, \widetilde{\overline{v}}_i\right]$ is an ε-approximation to the interval $[\underline{v}_i, \overline{v}_i]$.*

Definition 6. *By the* problem of computing the united solution set*, we mean the following problem:* given *an interval linear system (with possible solutions) and a rational number $\varepsilon > 0$,* compute *a ε-approximation to the interval hull $[\underline{y}_1, \overline{y}_1] \times \ldots \times [\underline{y}_m, \overline{y}_m]$ of the united solution set.*

In other words, for each j, we want to find ε-approximations to the following two values:

- the minimum $\underline{y}_j$ of all the values y_j corresponding to all possible tuples $(y_1, \ldots, y_j, \ldots, y_m)$, and
- the maximum $\overline{y}_j$ of all the values y_j corresponding to all possible tuples $(y_1, \ldots, y_j, \ldots, y_m)$.

What is known. It is known that the problem of computing the united solution set is, in general, NP-hard; see, e.g., [3]. Moreover, it is known that even the problem of checking whether there are any possible solutions is NP-hard [3].

What if all measurements have the same accuracy? The book [3] lists several proofs that the problem of computing the united solution set is NP-hard; each of these proofs uses intervals of different width – corresponding to situations when we measure different coefficients $a_{i,j}$ and b_i with different accuracy.

What if all the measurements have the same accuracy – i.e., all non-degenerate intervals have the same width?

On the one hand, in some such cases, it is possible to find a feasible algorithm for computing the united solution set; see, e.g., [2]. On the other hand, it was proven, in [1], that the problem of *checking* whether there are any possible solutions is still NP-hard even if we limit ourselves to measurements with the same accuracy.

What we do. In this paper, we show that for the cases when all the measurements have the same accuracy and the system has a possible solution (i.e., the united solution set is non-empty), the problem of *computing* the united solution set is also NP-hard.

2 Main Result

Definition 7. *Let $\Delta > 0$ be a rational number. We say that an interval linear system $\sum_{j=1}^{m} \boldsymbol{a}_{ij} \cdot y_j = \boldsymbol{b}_i$ is* uniformly Δ-accurate *if each interval $\boldsymbol{a}_{ij}$ or $\boldsymbol{b}_i$ is either identically 0 or has half-width Δ.*

Theorem 2.12. [1] *For every $\Delta > 0$, it is NP-hard to check whether a uniformly Δ-accurate interval linear system has a possible solution.*

Proposition. *For every $\Delta > 0$, the problem of computing the united solution for uniformly Δ-accurate interval linear systems is NP-hard.*

Comment. In other words, for every $\Delta > 0$, the following problem is NP-hard:

- *given:* a positive real number $\varepsilon > 0$ and a uniformly Δ-accurate interval linear system that has possible solutions;
- *compute:* an ε-approximation to the interval hull $[\underline{y}_1, \overline{y}_1] \times \ldots \times [\underline{y}_m, \overline{y}_m]$ of the united solution set.

Discission. It is important to mention that in general, the fact that it is NP-hard to check whether a system of equations *has* a solution does not necessarily mean that the problem of *computing* the solution when it exists is NP-hard.

As a simple example of such a situation, let us consider the following problem:

- *given:* a linear interval system with unknowns $y_1, \ldots, y_m$ in which one of the equations has the form $y_1 = 1$;
- *compute:* the set of all the values y_1 corresponding to all possible solutions $(y_1, \ldots, y_m)$ of this system.

In this case, checking whether this problem has a solution – i.e., whether the desired set if non-empty – is NP-hard. However, if we are limiting ourselves only to interval linear systems which are known to have possible solution, then the solution to this problem is trivial: for such systems, the desired set consists of a single value 1.

3 Proof of the Proposition

1°. By definition (see, e.g., [3, 5]), a problem P_0 is NP-hard if every problem from the class NP can be reduced to this problem P_0. Thus, to prove that a given problem P_g is NP-hard, it is sufficient to prove that a known NP-hard problem P_k can be reduced to P_g.

As such a problem P_k, we take the following *subset sum* problem (see, e.g., [3, 5]): given positive integers $s_1, \ldots, s_n$, find the values $\varepsilon_i \in \{-1, 1\}$ for which $\sum\limits_{i=1}^{n} \varepsilon_i \cdot s_i = 0$.

2°. To prove the Proposition, we will reduce each instance $(s_1, \ldots, s_n)$ of the subset sum problem to following interval linear system consisting of the following $p = 2n + 2$ equations with $m = n + 1$ unknowns $y_1, \ldots, y_n, y_{n+1}$:

- for $i \le n$, the corresponding equation takes the form
$$[\Delta, 3\Delta] \cdot y_i + [-\Delta, \Delta] \cdot y_{n+1} = 0; \tag{1}$$
- for $n + 1 \le i \le 2n$, the corresponding equation takes the form
$$[-\Delta, \Delta] \cdot y_{i-n} + [-3\Delta, -\Delta] \cdot y_{n+1} = 0; \tag{2}$$
- the equations corresponding to $i = 2n + 1$ and $i = 2n + 2$ have the form
$$[1, 1 + 2\Delta] \cdot y_{n+1} = [-\Delta, \Delta]; \tag{3}$$
$$\sum_{j=1}^{n} [M \cdot s_j - \Delta, M \cdot s_j + \Delta] \cdot y_j = 0, \tag{4}$$
where we denoted $M \stackrel{\text{def}}{=} 3\Delta \cdot n$.

One can easily check that this system is Δ-accurate.

Let us prove the following two implications:

- if the original instance of the subset sum has a solution, then the interval $\boldsymbol{y}_1$ (corresponding to the interval hull of the united solution set) is equal to $[-\Delta, \Delta]$;
- on the other hand, if the original instance of the subset problem does not have a solution, then $\boldsymbol{y}_1 = [0, 0]$.

If we compute the lower endpoint $\underline{y}_1$ of the interval $\boldsymbol{y}_1$ with accuracy $\varepsilon < \Delta/2$, we will get a rational number $\widetilde{\underline{y}}_1$ for which $|\underline{y}_1 - \widetilde{\underline{y}}_1| \le \varepsilon < \Delta/2$. Hence:

- if the original instance of the subset sum has a solution, then $\underline{y}_1 = -\Delta$ and thus, $\widetilde{\underline{y}}_1 < -\Delta/2$;
- on the other hand, if the original instance of the subset problem does not have a solution, then $\underline{y}_1 = 0$ and thus, $\widetilde{\underline{y}}_1 > -\Delta/2$.

Thus, if we could approximate $\underline{y}_1$ with accuracy ε, then, by comparing the resulting rational number $\widetilde{\underline{y}}_1$ with another rational number $-\Delta/2$, we would be able to tell whether a given instance of the subset problem has a solution. Therefore, we will have the desired reduction of the subset sum problem to our problem.

3°. To prove the above implications, let us first analyze the system (1)–(4).

For each $j \le n$, the fact that the tuple $(y_1, \ldots, y_{n+1})$ is a possible solution means, in particular, that the equation (1) is satisfied for $i = j$, i.e., that we have

$$a_{j,j} \cdot y_j + a_{j,n+1} \cdot y_{n+1} = 0$$

for some coefficients $a_{j,j} \in [\Delta, 3\Delta]$ and $a_{j,n+1} \in [-\Delta, \Delta]$. Thus, $y_j = r_j \cdot y_{n+1}$, where the coefficient $r_j \stackrel{\text{def}}{=} a_{j,n+1}/a_{j,j}$ takes a value from the interval $[-\Delta, \Delta]/[\Delta, 3\Delta] = [-1, 1]$. So, $|r_j| \le 1$.

Similarly, the equation (2) corresponding to $i = n + j$ means that

$$a_{n+j,j} \cdot y_j + a_{n+j,n+1} \cdot y_{n+1} = 0$$

for some coefficients $a_{n+j,j} \in [-\Delta, \Delta]$ and $a_{n+j,n+1} \in [\Delta, 3\Delta]$. Here, $|a_{n+j,j}| \le \Delta$ and $|a_{n+j,n+1}| \ge \Delta$. Substituting $y_j = r_j \cdot y_{n+1}$, with $|r_j| \le 1$, into this equation, we conclude that

$$(a_{n+j,j} \cdot r_j) \cdot y_{n+1} = (-a_{n+j,n+1}) \cdot y_{n+1}. \tag{5}$$

3.1°. We either have $y_{n+1} = 0$ or $y_{n+1} \ne 0$.

If $y_{n+1} = 0$, then from $y_j = r_j \cdot y_{n+1}$, we conclude that $y_j = 0$ for all $j \le n$. In this case, we have a tuple consisting of all zeros. One can check that this tuple is indeed a possible solution of the system (1)–(4).

3.2°. If $y_{n+1} \ne 0$, then, dividing both sides of the equation (5) by y_{n+1}, we conclude that

$$a_{n+j,j} \cdot r_j = -a_{n+j,n+1}. \tag{6}$$

Since $-a_{n+j,n+1} \ge \Delta$, we cannot have $a_{n+j,j} = 0$. If we had $|r_j| < 1$, then we would have $|a_{n+j,j} \cdot r_j| < |a_{n+j,j}| \le \Delta$, which contradicts to the fact that for $-a_{n+j,n+1} = a_{n+j,j} \cdot r_j$, we have $|-a_{n+j,n+1}| \ge \Delta$. Since $|r_j| \le 1$ and it is not possible to have $|r_j| < 1$, we conclude that $|r_j| = 1$, i.e., that $y_j = r_j \cdot y_{n+1}$ for some $r_j \in \{-1, 1\}$. Thus, all possible solutions $(y_1, \ldots, y_{n+1})$ with $y_{n+1} \ne 0$ have the form $y_j = \pm y_{n+1}$ for all $j \le n$.

4°. From the equation (3), it follows that $|y_{n+1}| \le \Delta$. Since $y_1 = r_1 \cdot y_{n+1}$ for $r_1 = \pm 1$, we conclude that $|y_1| \le \Delta$ for all possible solutions $(y_1, \ldots, y_{n+1})$.

5°. The equation (4) means that for some values α_j for which $|\alpha_j| \le \Delta$, we have

$$\sum_{j=1}^{n} (M \cdot s_j + \alpha_j) \cdot y_j = 0,$$

i.e.,

$$M \cdot \sum_{j=1}^{n} s_j \cdot y_i = -\sum_{j=1}^{n} \alpha_j \cdot y_j. \tag{7}$$

Substituting $y_j = r_j \cdot y_{n+1}$ into the formula (7) and dividing both sides by $y_{n+1} \neq 0$, we conclude that

$$M \cdot \sum_{j=1}^{n} r_j \cdot s_j = -\sum_{j=1}^{n} \alpha_j \cdot r_j. \tag{8}$$

Since $|\alpha_j| \le \Delta$ and $r_j = \pm 1$, we have

$$\left|\sum_{j=1}^{n} \alpha_j \cdot r_j\right| \le \sum_{j=1}^{n} |\alpha_j| \le n \cdot \Delta.$$

Thus, from (8), we get

$$M \cdot \left|\sum_{j=1}^{n} r_j \cdot s_j\right| \le n \cdot \Delta. \tag{9}$$

Dividing both sides of this inequality by $M = 3\Delta \cdot n$, we conclude that

$$\left|\sum_{j=1}^{n} r_j \cdot s_j\right| \le \frac{1}{3}. \tag{10}$$

The values s_j are integers, the values $r_j = \pm 1$ are also integers, so the sum $\sum\limits_{j=1}^{n} r_j \cdot s_j$ is also an integer. The fact that the absolute value of this integer does not exceed 1/3 means that this integer is equal to 0, i.e., that $\sum\limits_{j=1}^{n} r_j \cdot s_j = 0$.

Thus, if the system (1)–(4) has a non-zero possible solution, then the original instance of the subset problem has a solution.

6°. From the previous phrase, we can conclude that if the original instance of the subset problem has no solutions, then the system (1)–(4) cannot have non-zero solutions. This, in this case, the only possible solution to the system (1)–(4) is an all-zeros solution.

In this case, the interval $\boldsymbol{y}_1$ is equal to $[0, 0]$.

7°. If the original instance of the subset sum problem has a solution $\varepsilon_i \in \{-1, 1\}$ for which $\sum\limits_{i=1}^{n} \varepsilon_i \cdot s_i = 0$, then, for each value $y_1 \in [-\Delta, \Delta]$, we can take $y_i = \dfrac{\varepsilon_i}{\varepsilon_1} \cdot y_1$ for all $i \le n$ and $y_{n+1} = \dfrac{y_1}{\varepsilon_1}$. Let us show that these values form a possible solution of the system (1)–(4); indeed:

- Each equation of the type (1) is satisfied since for the selected values y_j, we have

  $$\Delta \cdot y_i + (-\Delta \cdot \varepsilon_i) \cdot y_{n+1} = 0,$$

  with $\Delta \in [\Delta, 3\Delta]$ and $-\Delta \cdot \varepsilon_i \in [-\Delta, \Delta]$.

- Each equation of type (2) is satisfied since for $i = n + j$, we have

  $$(\Delta \cdot \varepsilon_i) \cdot y_i + (-\Delta) \cdot y_{n+1} = 0,$$

  with $\Delta \cdot \varepsilon_i \in [-\Delta, \Delta]$ and $-\Delta \in [-3\Delta, -\Delta]$.

- The equation (3) is satisfied since

$$1 \cdot y_{n+1} = y_{n+1},$$

where $1 \in [1, 1 + 2\Delta]$ and $y_{n+1} \in [-\Delta, \Delta]$.

- Finally, the equation (4) is satisfied since we have

$$\sum_{j=1}^{n} (M \cdot s_j) \cdot y_j = 0,$$

with $M \cdot s_i \in [M \cdot s_i - \Delta, M \cdot s_i + \Delta]$.

On the other hand, we know that for all possible solutions, we have $|y_1| \le \Delta$. Thus, in this case, the desired interval $\boldsymbol{y}_1$ is equal to $[-\Delta, \Delta]$.

The reduction is proven, and so is the Proposition.

Comment. In the above reductions, the number of equations is, in general larger than the number of unknowns; however, we can easily make these two numbers equal if we add extra unknowns that do not affect equations at all. Thus, the problem remains NP-hard even if we limit ourselves to square systems.

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