A Fuzzy Set Estimation Using Interval Contractors: Application to Localization*

J. Tillet
Lab-STICC, ENSTA Bretagne, 2 rue François Verny, 29806 Brest Cedex 9, France
joris.tillet@ensta-bretagne.org

L. Jaulin
Lab-STICC, ENSTA Bretagne, 2 rue François Verny, 29806 Brest Cedex 9, France
luc.jaulin@ensta-bretagne.fr

F. Le Bars
Lab-STICC, ENSTA Bretagne, 2 rue François Verny, 29806 Brest Cedex 9, France
fabrice.le_bars@ensta-bretagne.fr

R. Boukezzoula
Laboratoire d'Informatique, Systèmes, Traitement de l'Information et de la Connaissance (LISTIC) Group – Polytech'Savoie, BP 80439, 74944 Annecy le Vieux Cedex, France
reda.boukezzoula@univ-smb.fr

Abstract

This paper proposes a new approach to characterize fuzzy sets defined by membership functions using interval analysis. The goal is to combine pieces of information (the granules) to build a fuzzy set providing a representation of the knowledge we have on the parameter vector we want to estimate. Then the different α-cuts of this fuzzy set can be approximated by an interval procedure. The proposed formalism can handle

*Submitted: September 14, 2021; Revised: February 7, 2022; Accepted: February 7, 2022.

efficiently various situations since there is a lot of freedom to define the wanted combination. The information contained in a random vector is represented by a membership function, which is issued from the composition of a score function and characteristic functions associated with some elementary epistemic sets, considered as the elementary granules of knowledge. Each granule is attached to a given measurement or any other elementary information we have on the vector to be estimated. The score function is an aggregation operator that defines the weighting of these granules, simplifying their combination. Thus, it is possible to deal with complex outliers-related situations in a context where uncertainties are only partially known. The proposed approach makes it possible to obtain an efficient interval-based algorithm able to find an inner and an outer approximation of the α-cut to be characterized. An application related to the localization of an underwater robot is presented to illustrate the efficiency of the approach.

Keywords: α-cut characterization, fuzzy set estimation, interval contractors, underwater localization
AMS subject classifications: 65G20, 65G30, 65G40

1 Introduction

When dealing with localization problems, the mainstream approaches to represent uncertainties are based on probability theory [46]. There exist many efficient probabilistic estimators to characterize continuous quantities such as positions, orientations or trajectories [30, 48]. Probabilistic methods require a complete representation of all errors. Often, it is not possible, and they need to add some unrealistic assumptions such as the independence between errors to define and characterize the posterior density function of the parameters to be estimated [1, 13, 44]. As a result, we may obtain some results that may lack integrity, *i.e.*, the resulting confidence region is over-optimistic and may be far from the actual parameter vector. In such a context, one should move toward more flexible methods that can relax the strict requirements on the statistical properties of the manipulated data.

Set membership methods [18, 25, 32, 40] are particularly attractive to deal with situations where the representation of the uncertainties and the correlations between them are not well known. They can benefit from set membership tools such as interval analysis [35] and contractor programming [12, 42] to characterize the set of all parameters that are consistent with all the data. The corresponding interval estimators have been shown very reliable on many different types of localization problems [4, 14, 17, 31, 41, 43]. The interval approach can be considered as relatively poor and less specific than the probabilistic methods. Nevertheless, it can address insufficient data, allow more flexible manipulations in complex and non-linear computations by avoiding strict statistical requirements that are not always verified in practice.

To remedy the lack of specificity of set membership methods and permit the representation of the progressive belonging, fuzzy sets [21], possibility theory [22] or belief theory [15, 16] can be envisioned. They add a vertical dimension to exhibit the distribution of the variables. These distributions are not *probability distributions* but *possibility distributions.*

These conceivable approaches, mainly used when the variables are discrete, are particularly suited to deal with estimation problems where the uncertainty cannot be completely modeled. When the variables are continuous, *i.e.*, vectors of $\mathbb{R}^n$, interval methods can still be used with a graduality, see, *e.g.*, [36] in the context of belief functions, [5, 6, 7] for fuzzy estimation, [20] in the context of possibility theory or [19, 37, 38] when combined with probabilistic methods.

The main contribution of this paper is a new formulation of the α-cuts principle, which allows us to use existing interval-based algorithms for the resolution of non-linear parameter estimation problems. The principle of the resulting method is to apply interval arithmetic on boxes that cover the parameter space. If all vectors in a given box satisfy a required level of consistency, the box is considered as inside the solution set. If for the whole box, this level cannot be reached, the box is rejected. If we do not have the same conclusion for all vectors of the box, the box is bisected into two boxes which are added to the list of covering boxes. The formulation we consider for testing the boxes can be understood as an extension of the *t-norm* fuzzy logic [23] which has been introduced to generalize classical two-valued logic by admitting intermediary truth values between 1 (truth) and 0 (false), representing degrees of the truth of propositions.

The paper is organized as follows. Section 2 provides a formulation for the α-cut of a membership function that is piecewise constant. This choice will allow us to fit with existing interval algorithms that will be used for the resolution. Section 3 presents the interval-based approach for the characterization of the α-cuts. Section 4 shows an application to the localization of an underwater robot equipped with a sonar. Moreover, an experiment involving a real underwater robot inside a swimming pool is treated. Section 5 concludes the paper.

2 A New Formulation of the α-cuts Principle

This section presents the notion of score function, which can be interpreted as an m-ary extension of a t-norm [23]. It belongs to the class of aggregation operators for fuzzy sets used for information fusion [24, 29, 45]. Score functions will be used to build membership functions classically used to represent fuzzy sets [49]. Then we will take the associated α-cut to get a set enclosure of some random vectors that have to be estimated. To allow us to use efficient interval algorithms, we will introduce a specific class of fuzzy sets that are defined by piecewise membership functions. These functions are built from the composition of a score function σ with characteristic functions of sets $\mathbb{Z}_j, j \in \{1, \ldots, m\}$. The sets $\mathbb{Z}_j$ are defined by inequalities and are called *the granules.*

2.1 Definitions

The *characteristic function* associated with a subset $\mathbb{A}$ of $\mathbb{R}^n$ is the function $\chi_{\mathbb{A}} : \mathbb{R}^n \to \{0, 1\}$ such that $\chi_{\mathbb{A}}(\mathbf{x}) = 1$ if $\mathbf{x} \in \mathbb{A}$ and $\chi_{\mathbb{A}}(\mathbf{x}) = 0$ otherwise. We define a *score function* σ as a function from $\{0, 1\}^m$ to $[0, 1]$ which satisfies the following properties:

$$\begin{array}{ll} \text{(i)} & \sigma(0, 0, \ldots, 0) = 0 \\ \text{(ii)} & \sigma(1, 1, \ldots, 1) = 1 \\ \text{(iii)} & \forall j, a_j \leq b_j \Rightarrow \sigma(a_1, \ldots, a_m) \leq \sigma(b_1, \ldots, b_m) \\ & \qquad\qquad\qquad\qquad\qquad\qquad \text{(monotonicity)} \end{array} \tag{1}$$

We consider fuzzy sets $\mathbb{X}$ with a membership function $\mu_{\mathbb{X}}(\mathbf{x})$ which can be defined as follow:

$$\mu_{\mathbb{X}} : \left\{ \begin{array}{ccc} \mathbb{R}^n & \rightarrow & [0,1] \\ \mathbf{x} & \rightarrow & \sigma\left(\zeta^1(\mathbf{x}), \ldots, \zeta^m(\mathbf{x})\right) \end{array} \right. \tag{2}$$

where $\zeta^1, \ldots, \zeta^m$ are the characteristic functions of the m sets $\mathbb{Z}_1, \ldots, \mathbb{Z}_m$ of $\mathbb{R}^n$, *i.e.*, $\zeta^j(\mathbf{x}) = \chi_{\mathbb{Z}_j}(\mathbf{x})$. The sets $\mathbb{Z}_1, \ldots, \mathbb{Z}_m$ can be seen as knowledge granules we have on $\mathbf{x}$. These *granules* translate an information such as a measurement or a constraint on $\mathbf{x}$.

Example 2.1 *To illustrate the approach, suppose that we have four ring-shaped granules, $\mathbb{Z}_1, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4$, corresponding to the position for a robot consistent with some measured distances d_j to different landmarks $\mathbf{m}(j) = (m_1(j), m_2(j))$. More precisely, we have*

$$\mathbb{Z}_j = \left\{ \mathbf{x} \in \mathbb{R}^2 \mid \sqrt{(x_1 - m_1(j))^2 + (x_2 - m_2(j))^2} \in [d_j] \right\} \tag{3}$$

where the intervals $[d_j]$ and the landmark coordinates $\mathbf{m}(j)$ are given by Table 1.

Table 1: Measured distances and coordinates of the landmarks.

j	1	2	3	4
$[d_j]$	$[2.2, 4.2]$	$[4.4, 6.4]$	$[7.1, 9.1]$	$[4.1, 6.1]$
$\mathbf{m}(j)$	$(-1, 3)$	$(5, 2)$	$(8, -1)$	$(1, -5)$

We define the fuzzy set $\mathbb{X}$ by its set membership function

$$\mu_{\mathbb{X}}(\mathbf{x}) = \frac{\zeta^1(\mathbf{x}) + 2\zeta^2(\mathbf{x}) + \zeta^3(\mathbf{x}) + \zeta^4(\mathbf{x})}{5} \tag{4}$$

where $\zeta^j(\mathbf{x})$ are the characteristic functions of the granules $\mathbb{Z}_j$. The expression for $\mu_{\mathbb{X}}(\mathbf{x})$ translates the fact that the position of the robot should satisfy the measured distance intervals such as the confidence or the reliability associated with $\mathbf{x} \in \mathbb{Z}_2$ is twice the reliability of $\mathbf{x} \in \mathbb{Z}_j, j \neq 2$. The fuzzy set $\mathbb{X}$ can be represented by its α-cuts $\mathbb{X}_\alpha$, $\alpha \in \{\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1\}$ as illustrated by Figure 1.

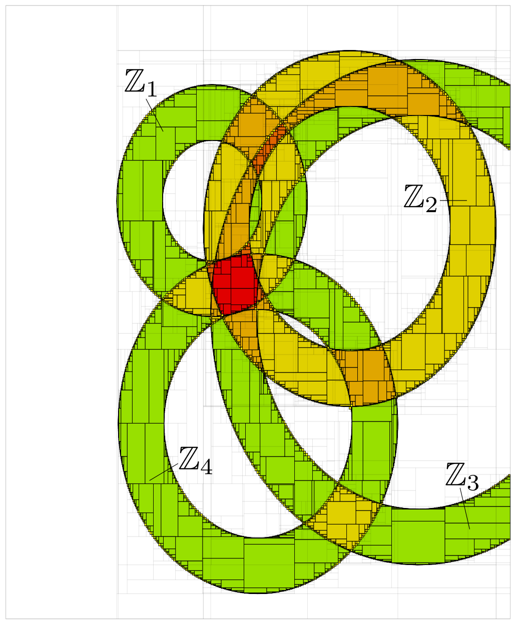
Figure 1 has been obtained by using the interval method that will be seen later in Section 3. We observe that the function μ is made with plateaus which are due to the specific form of our function μ. The boxes that are visible in Figure 1a are due to the interval method, which bisects and tests interval values for x_1 and x_2.

The corresponding Python program can be downloaded and executed online at the following link:

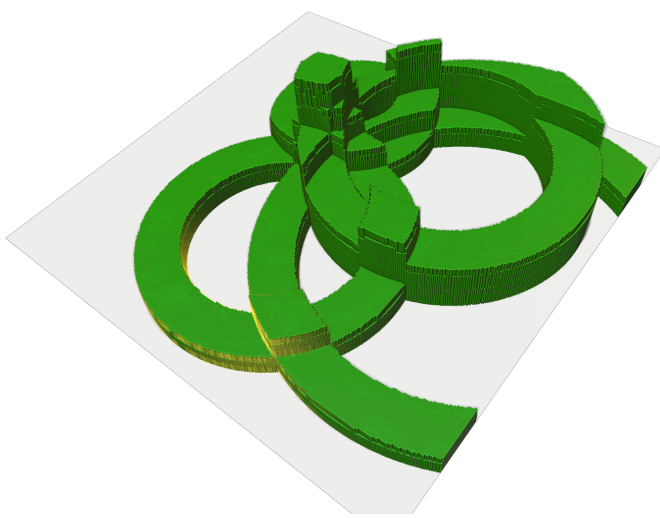
https://replit.com/@TilletJ/Alpha-cut-characterization.

2.2 Usefulness of the Proposed α-cuts Approach

Membership functions are often used to represent fuzzy sets [21]. Functions μ that have the form given by (2) are called *piecewise constant membership functions.* Characterizing such a function amounts to characterizing a finite number of its α-cuts. Now, we will show that these α-cuts are sets defined by inequalities, and thus a set-inversion algorithm can efficiently compute inner and outer approximations, as we have illustrated in Example 2.1. Let us now recall some definitions.



(a) Top view with α going from green to red



(b) 3D view

Figure 1: Representation of the fuzzy set $\mathbb{X}$ with its vertical dimension. (a): The value of α goes from $\frac{1}{5}$ (green) to 1 (red). (b): 3D view of $\mathbb{X}$ in the (x_1, x_2, α)-space.

An α-cut of a fuzzy set $\mathbb{X}$ is a *crisp* set which can be defined as

$$\begin{aligned}\mathbb{X}_\alpha &= \{\mathbf{x} \mid \mu_{\mathbb{X}}(\mathbf{x}) \geq \alpha\} \\ &= \left\{\mathbf{x} \mid \sigma\left(\zeta^1(\mathbf{x}), \ldots, \zeta^m(\mathbf{x})\right) \geq \alpha\right\},\end{aligned} \tag{5}$$

where α is called the *membership degree*. The degree α gives us the vertical dimension on the representation of uncertainty (degrees of confidence, of reliability, of flexibility, ...). In our formalism, the number of values that can be taken by $\mu_{\mathbb{X}}(\mathbf{x})$ is finite and smaller than 2^m, the number of values that can be taken by the vector $\left(\zeta^1(\mathbf{x}), \ldots, \zeta^m(\mathbf{x})\right)$.

Example 2.2 *Table 2 provides some examples of membership functions, threshold α and the corresponding α-cuts.*

Table 2: Examples of membership functions with threshold α and their corresponding α-cuts.

	$\mu_{\mathbb{X}}(\mathbf{x})$	α	$\mathbb{X}_\alpha$
(i)	$\frac{\zeta^1(\mathbf{x})+\zeta^2(\mathbf{x})}{2}$	1	$\mathbb{Z}_1 \cap \mathbb{Z}_2$
(ii)	$\frac{\zeta^1(\mathbf{x})+\zeta^2(\mathbf{x})}{2}$	0.5	$\mathbb{Z}_1 \cup \mathbb{Z}_2$
(iii)	$\frac{\zeta^1(\mathbf{x})+2\zeta^2(\mathbf{x})+\zeta^3(\mathbf{x})+\zeta^4(\mathbf{x})}{5}$	0.5	$(\mathbb{Z}_1 \cap \mathbb{Z}_2) \cup (\mathbb{Z}_2 \cap \mathbb{Z}_4) \cup (\mathbb{Z}_2 \cap \mathbb{Z}_3)$ $\cup (\mathbb{Z}_1 \cap \mathbb{Z}_3 \cap \mathbb{Z}_4)$
(iv)	$\frac{1}{m}\sum_j \zeta^j(\mathbf{x})$	$1 - \frac{q}{m}$	$\bigcap^{\{q\}} \mathbb{Z}_j$
(v)	$\min\left(\zeta^1(\mathbf{x}), \frac{1}{m}\sum_j \zeta^j(\mathbf{x})\right)$	$1 - \frac{q}{m}$	$\mathbb{Z}_1 \cap \bigcap^{\{q\}} \mathbb{Z}_j$

Line (i) should be understood as follows:

$$\begin{array}{rcl}\frac{\zeta^1(\mathbf{x})+\zeta^2(\mathbf{x})}{2} \geq 1 & \Leftrightarrow & \zeta^1(\mathbf{x}) = 1 \textit{ and } \zeta^2(\mathbf{x}) = 1 \\ & \Leftrightarrow & \mathbf{x} \in \mathbb{Z}_1 \textit{ and } \mathbf{x} \in \mathbb{Z}_2 \\ & \Leftrightarrow & \mathbf{x} \in \mathbb{Z}_1 \cap \mathbb{Z}_2.\end{array} \tag{6}$$

In (iv) and (v), $\bigcap^{\{q\}} \mathbb{Z}_j$ denotes the relaxed intersection *[28],* i.e., *the set of all $\mathbf{x}$ which belong to all sets $\mathbb{Z}_j$ except q of them.*

The general procedure to translate the inequality $\mu_{\mathbb{X}}(\mathbf{x}) \geq \alpha$ into a set expression is illustrated by Figure 2 on case (iii). We first draw the Karnaugh table associated to $\mu_{\mathbb{X}}(\mathbf{x}) \geq \alpha$. Then, we form the Karnaugh blocks in order to write the Disjunctive Normal Form. In the example, we get

$$\mathbb{X}_\alpha = (\mathbb{Z}_1 \cap \mathbb{Z}_2) \cup (\mathbb{Z}_2 \cap \mathbb{Z}_4) \cup (\mathbb{Z}_2 \cap \mathbb{Z}_3) \cup (\mathbb{Z}_1 \cap \mathbb{Z}_3 \cap \mathbb{Z}_4). \tag{7}$$

Note that some factorizations could yield a shorter expression, e.g.*,*

$$\mathbb{X}_\alpha = \mathbb{Z}_2 \cap (\mathbb{Z}_1 \cup \mathbb{Z}_3 \cup \mathbb{Z}_4) \quad \cup \quad (\mathbb{Z}_1 \cap \mathbb{Z}_3 \cap \mathbb{Z}_4). \tag{8}$$

Unfortunately, depending on $\mu_{\mathbb{X}}(\mathbf{x})$, the procedure may yield an expression with a length that is exponential in m even after factorization routines. This is why we prefer here

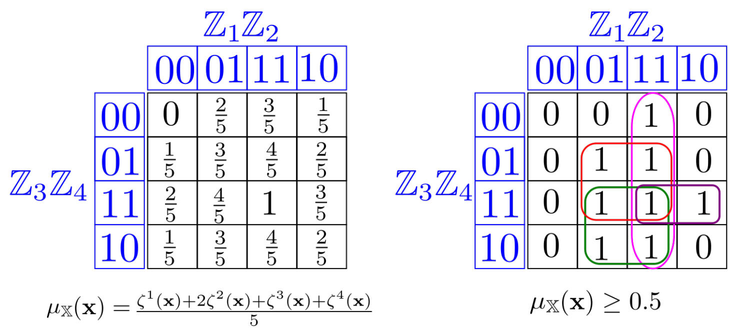


Figure 2: Left: function $\mu_{\mathbb{X}}(\mathbf{x})$; Right: Karnaugh table associated to the constraint $\mu_{\mathbb{X}}(\mathbf{x}) \geq \alpha$.

to work directly on the expression of $\mu_{\mathbb{X}}(\mathbf{x})$ and not on the set expression. Since the expression for μ admits more symbols ($+, -, \exp, \sin, \max, \min, \dots$) than for the expression for $\mathbb{X}_\alpha$ ($\cup, \cap, \overline{\ }$), we limit the combinatorial length and the corresponding evaluation. This is one of the main contributions of this paper.

2.3 Complementary of an α-cut

This section gives a transformation which will allow us to define the complementary of α-cuts of a piecewise constant membership function as the α-cuts of some other similar membership functions, *i.e.*, with the form (2). From this transformation, we derive a procedure to compute an inner approximation of the α-cuts. Computing an inner approximation is important to get efficient interval algorithms. Indeed, these interval algorithms are close to the branch and bound design, and thus require the two bounds of the set approximation to converge quickly [3, 39]. For instance, in Figure 1a, all boxes that have been represented have been shown to be inside an α-cut and outside another one. The algorithm used is Sivia (Set Inversion Via Interval Analysis) [27]. As an example, the green boxes are inside the $\frac{1}{5}$-cut and outside the $\frac{2}{5}$-cut. If we have no procedure to prove that the green boxes were inside the $\frac{1}{5}$-cut, we would have to bisect them. The number of boxes would have been huge for the same accuracy. The computing burden is thus considerably reduced by an inner test.

The goal of this section is to show that the same procedure can be used to prove that a box is inside an α-cut or to prove that it is outside it.

Proposition 2.1 (De Morgan rule) . *The complementary set of the α-cut $\mathbb{X}_\alpha = \left\{\mathbf{x} \mid \sigma\left(\zeta^1(\mathbf{x}), \dots, \zeta^m(\mathbf{x})\right) \geq \alpha\right\}$ is the α-cut defined by*

$$\overline{\mathbb{X}}_\alpha = \left\{\mathbf{x} \mid \bar{\sigma}\left(\bar{\zeta}^1(\mathbf{x}), \dots, \bar{\zeta}^m(\mathbf{x})\right) > \bar{\alpha}\right\} \tag{9}$$

where

$$\begin{aligned} \bar{\zeta}^j &= 1 - \zeta^j \\ \bar{\sigma}\left(a_1, \dots, a_m\right) &= 1 - \sigma\left(1 - a_1, \dots, 1 - a_m\right) \\ \bar{\alpha} &= 1 - \alpha. \end{aligned} \tag{10}$$

Proof: First, let us note that since the function $\bar{\sigma}$ only takes a finite number of values, the strict inequality ($>$) can always be transformed into a non-strict inequality ($\geq$). Since $\mathbb{X}_\alpha = \left\{\mathbf{x} \mid \sigma\left(\zeta^1(\mathbf{x}), \ldots, \zeta^m(\mathbf{x})\right) \geq \alpha\right\}$, we have

$$\begin{aligned}\mathbf{x} \in \overline{\mathbb{X}}_\alpha &\Leftrightarrow \sigma\left(\zeta^1(\mathbf{x}), \ldots, \zeta^m(\mathbf{x})\right) < \alpha \\ &\Leftrightarrow 1 - \sigma\left(\zeta^1(\mathbf{x}), \ldots, \zeta^m(\mathbf{x})\right) > 1 - \alpha \\ &\Leftrightarrow 1 - \sigma\left(1 - \bar{\zeta}^1(\mathbf{x}), \ldots, 1 - \bar{\zeta}^m(\mathbf{x})\right) > \bar{\alpha} \\ &\Leftrightarrow \bar{\sigma}\left(\bar{\zeta}^1(\mathbf{x}), \ldots, \bar{\zeta}^m(\mathbf{x})\right) > \bar{\alpha}.\end{aligned}$$

We now have to check that $\bar{\sigma}$ is a membership function (see (1)).
(i) We have $\bar{\sigma}(0, \ldots, 0) = 1 - \sigma(1-0, \ldots, 1-0) = 0$.
(ii) We have $\bar{\sigma}(1, \ldots, 1) = 1 - \sigma(1-1, \ldots, 1-1) = 1$.
(iii) We now check the monotonicity of $\bar{\sigma}$

$$\begin{aligned}&(a_1, \ldots, a_m) \leq (b_1, \ldots, b_m) \\ \Rightarrow &(1-a_1, \ldots, 1-a_m) \geq (1-b_1, \ldots, 1-b_m) \\ \Rightarrow &\sigma(1-a_1, \ldots, 1-a_m) \geq \sigma(1-b_1, \ldots, 1-b_m) \\ \Leftrightarrow &1 - \sigma(1-a_1, \ldots, 1-a_m) \leq 1 - \sigma(1-b_1, \ldots, 1-b_m) \\ \Leftrightarrow &\bar{\sigma}(a_1, \ldots, a_m) \leq \bar{\sigma}(b_1, \ldots, b_m).\end{aligned}$$

□

Example 2.3 *Consider again the set* $\mathbb{X}_\alpha = \mathbb{Z}_2 \cap (\mathbb{Z}_1 \cup \mathbb{Z}_3 \cup \mathbb{Z}_4) \cup (\mathbb{Z}_1 \cap \mathbb{Z}_3 \cap \mathbb{Z}_4)$ *(see (8)) and defined by*

$$\frac{\zeta^1(\mathbf{x}) + 2\zeta^2(\mathbf{x}) + \zeta^3(\mathbf{x}) + \zeta^4(\mathbf{x})}{5} \geq 0.5. \tag{11}$$

Note that the corresponding score function is the same as in Example 2.1. The complementary set $\overline{\mathbb{X}}_\alpha$ *is defined by*

$$\begin{aligned}&1 - \frac{1-\bar{\zeta}^1(\mathbf{x})+2\left(1-\bar{\zeta}^2(\mathbf{x})\right)+1-\bar{\zeta}^3(\mathbf{x})+1-\bar{\zeta}^4(\mathbf{x})}{5} > 1 - 0.5 \\ \Leftrightarrow \quad &\frac{5-1+\bar{\zeta}^1(\mathbf{x})-2+2\bar{\zeta}^2(\mathbf{x})-1+\bar{\zeta}^3(\mathbf{x})-1+\bar{\zeta}^4(\mathbf{x})}{5} > 0.5 \\ \Leftrightarrow \quad &\frac{\bar{\zeta}^1(\mathbf{x})+2\bar{\zeta}^2(\mathbf{x})+\bar{\zeta}^3(\mathbf{x})+\bar{\zeta}^4(\mathbf{x})}{5} \geq 0.5.\end{aligned}$$

The following example illustrates a situation where the relaxed intersection is involved.

Example 2.4 *Consider again the set* $\mathbb{Z}_1 \ \cap \ \bigcap^{\{q\}} \mathbb{Z}_j$ *(see Example 2.2) defined by*

$$min\left(\zeta^1(\mathbf{x}), \frac{1}{m}\sum_j \zeta^j(\mathbf{x})\right) \geq 1 - \frac{q}{m}. \tag{12}$$

The complementary set is defined by

$$
\begin{aligned}
1 - min\left(1-\bar{\zeta}^1(\mathbf{x}), \frac{1}{m}\sum_j \left(1-\bar{\zeta}^j(\mathbf{x})\right)\right) \\
> 1 - \left(1 - \frac{q}{m}\right) \\
\Leftrightarrow 1 - min\left(1-\bar{\zeta}^1(\mathbf{x}), 1 - \frac{1}{m}\sum_j \bar{\zeta}^j(\mathbf{x})\right) > \frac{q}{m} \\
\Leftrightarrow 1 + max\left(-1+\bar{\zeta}^1(\mathbf{x}), -1 + \frac{1}{m}\sum_j \bar{\zeta}^j(\mathbf{x})\right) > \frac{q}{m} \\
\Leftrightarrow max\left(\bar{\zeta}^1(\mathbf{x}), \frac{1}{m}\sum_j \bar{\zeta}^j(\mathbf{x})\right) \geq \frac{q+1}{m}.
\end{aligned}
$$

In this section, we have shown that the α-cuts (as well as their complementaries) of a specific class of fuzzy sets can be defined by inequalities with a specific form. This form involves a score function and characteristic functions of granules. The next section will exploit this particular form to characterize the fuzzy sets efficiently through their α-cuts.

3 Resolution Using Interval Contractors

In the previous section, we have shown how an α-cut of a specific class of fuzzy sets can be defined by an inequality involving a function which is piecewise constant. We will now show how we can build efficient interval contractors that will allow an inner and an outer characterization of the solution set.

3.1 Contractors

An *interval* $[x]$ of $\mathbb{R}$ is a closed connected set of $\mathbb{R}$ defined by its lower bound x^- and its upper bound x^+. A box $[\mathbf{x}]$ of $\mathbb{R}^n$ is the Cartesian product of n intervals. The set of all boxes of $\mathbb{R}^n$ is denoted by $\mathbb{IR}^n$. Note that an arithmetic exists on intervals allowing to use operators and functions such the division or the sine function (see *e.g.* [2, 35]). We now provide a definition of *contractor* associated to a set $\mathbb{X}$. It can be interpreted as an operator which takes a box $[\mathbf{x}]$ as input and contracts it without removing a single point of $\mathbb{X}$. The formal definition is the following.

A *contractor* $\mathcal{C}$ [11] for the set $\mathbb{X} \subset \mathbb{R}^n$ is an operator $\mathbb{IR}^n \mapsto \mathbb{IR}^n$ such that

$$
\begin{array}{ll}
\mathcal{C}([\mathbf{x}]) \subset [\mathbf{x}] & \text{(contractance)} \\
[\mathbf{x}] \cap \mathbb{X} \subset \mathcal{C}([\mathbf{x}]) & \text{(consistency)} \\
[\mathbf{x}] \subset [\mathbf{y}] \ \Rightarrow\ \mathcal{C}([\mathbf{x}]) \subset \mathcal{C}([\mathbf{y}]) & \text{(monotonicity)}
\end{array}
\tag{13}
$$

Figure 3 illustrates this definition. The boxes $[\mathbf{a}]$ and $[\mathbf{b}]$ are contracted into $\mathcal{C}([\mathbf{a}])$ and $\mathcal{C}([\mathbf{b}])$ and no points of $\mathbb{X}$ have been removed. Note that in our situation, we have $\mathcal{C} \circ \mathcal{C}([\mathbf{b}]) = \emptyset$. Indeed, the contractor $\mathcal{C}$ is not efficient enough to show in one step that $[\mathbf{b}]$ is outide $\mathbb{X}$ and has to be called twice to reach this conclusion. Let us stress that contractors are implemented with numerical algorithm involving floating-point numbers. To have the consistency property means that we have to check all

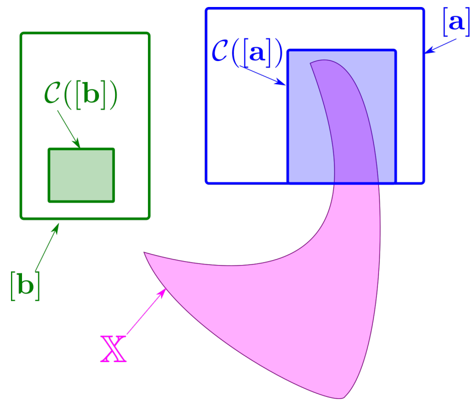


Figure 3: The boxes $[\mathbf{a}]$ and $[\mathbf{b}]$ are contracted into $\mathcal{C}([\mathbf{a}])$ and $\mathcal{C}([\mathbf{b}])$.

points in the input boxes. That can be done efficiently and rigorously using interval computations even if the number of points in the box is infinite and uncountable. As soon as a contractor is available for the set $\mathbb{X}$, we can obtain an approximation of $\mathbb{X}$ under the form of a paving (union of boxes) [26]. An α-cut can be expressed in terms of unions and intersections of sets. Interval contractors have been shown to be powerful in this context [47] through what is called a *contractor algebra* [11]. In what follows, we show that the use of membership functions will allow us to be more general and more flexible than using existing contractor algebra.

3.2 Contractors for α-cuts

Consider the α-cut expression of Eq. 5, where $\zeta^j(\mathbf{x}) = \chi_{\mathbb{Z}_j}$ are characteristic functions of some sets $\mathbb{Z}_j$, $j \in \{1, \ldots, m\}$ and σ is a score function. The problem to be considered now, is to find an efficient contractor for $\mathbb{X}_\alpha$. Three cases can be distinguished:

- **Interval**. The granules $\mathbb{Z}_j$ are intervals $[z](j)$ in Subsection 3.2.1;
- **Box**. The $\mathbb{Z}_j$ are boxes $[\mathbf{z}](j)$ in Subsection 3.2.2;
- **General**. The $\mathbb{Z}_j$ are any subsets of $\mathbb{R}^n$ for which a contractor is available in Subsection 3.2.3.

3.2.1 Interval-based α-cuts

Consider the problem of finding a contractor for $\mathbb{X}_\alpha$ in the case where the granules $\mathbb{Z}_j$ are intervals $[z](j)$ of $\mathbb{R}$. We want to compute the smallest interval that encloses all $x \in [x]$ such that $x \in \mathbb{X}_\alpha$ (see (5)). Note that computing this smallest interval amounts to finding the optimal contractor for the set $\mathbb{X}_\alpha$, *i.e.* a contractor $\mathcal{C}$ such that for all contractor $\mathcal{C}'$ for the set $\mathbb{X}_\alpha$ and for all interval $[x]$ we have $\mathcal{C}([x]) \subset \mathcal{C}'([x])$. We denote this contractor by $\mathcal{C}^{\text{interval}}_{\sigma,\alpha}$ since the score function σ and the scalar α are sufficient to define the set $\mathbb{X}_\alpha$, from the knowledge of characteristic functions ζ^j. We now describe a procedure (taken from [34]) which solves the problem with a complexity of $O(m \cdot \log(m))$. In what follows, $[z](1:m)$ denotes the list of intervals $\{[z](1), \ldots, [z](m)\}$.

Algorithm 1. Optimal contractor for the constraint $\sigma\left(\zeta^1(x),\dots,\zeta^m(x)\right)\geq\alpha$, when the $[z](i)$ are intervals.

	Algorithm $\mathcal{C}^{\text{interval}}_{\sigma,\alpha}$(in:$[z](1:m)$; inout:$[x]$)
1	Store all endpoints of $[z](1),\dots,[z](m)$ and $[x]$ inside a list $\mathcal{L}$
2	Remove elements of $\mathcal{L}$ that are not inside $[x]$
3	Sort $\mathcal{L}$ in ascending order
4	Take the smallest element a of $\mathcal{L}$ such that $\sigma(\zeta^1(a),\dots,\zeta^m(a))\geq\alpha$
5	If no a has been found, return $\emptyset$
6	Take the greatest element b of $\mathcal{L}$ such that $\sigma(\zeta^1(b),\dots,\zeta^m(b))\geq\alpha$
7	Return $[a,b]$.

3.2.2 Box-based α-cuts

Consider the problem of finding a contractor for $\mathbb{X}_\alpha$ in the case where the $\mathbb{Z}_j$ are boxes $[\mathbf{z}](j)$, $j\in\{1,\dots,m\}$ instead of intervals. Finding the optimal contractor for $\mathbb{X}_\alpha$ (see (5)) is known to be an NP hard problem [8, 34]. In order to build an efficient contractor for $\mathbb{X}_\alpha$, denoted by $\mathcal{C}^{\text{box}}_{\sigma,\alpha}$, we follow the procedure described in [9]. For this, we will project the problem with respect to each of the n axes and then call n times the contractor $\mathcal{C}^{\text{interval}}_{\sigma,\alpha}$ described in the previous subsection. The procedure will be based on the following proposition.

Proposition 3.1 *Consider m boxes $[\mathbf{z}](1),\dots,[\mathbf{z}](m)$. If σ is a score function and $i\in\{1,\dots,n\}$, we have*

$$\sigma\left(\zeta^1(\mathbf{x}),\dots,\zeta^m(\mathbf{x})\right)\leq\sigma\left(\zeta^1_i(x_i),\dots,\zeta^m_i(x_i)\right), \tag{14}$$

where $\zeta^j(\mathbf{x})$ and $\zeta^j_i(x_i)$ are the characteristic functions of $[\mathbf{z}](j)$ and $[z_i](j)$, respectively.

Proof: Consider a vector $\mathbf{x}=(x_1,\dots,x_n)$ and a box $[\mathbf{z}]=[z_1]\times\ldots\times[z_n]$ with characteristic function ζ. We have

$$\zeta(\mathbf{x})=\zeta_1(x_1)\cdot\zeta_2(x_2)\cdot\ldots\cdot\zeta_n(x_n) \tag{15}$$

and thus for all i, $\zeta(\mathbf{x})\leq\zeta_i(x_i)$. Since σ is monotonic, we get the inequality to be proved. □

Example 3.1 *Consider the situation of Figure 4 where the membership function is*

$$\mu_{\mathbb{X}}(\mathbf{x})=\sigma\left(\zeta^1(\mathbf{x}),\zeta^2(\mathbf{x}),\zeta^3(\mathbf{x})\right)=\frac{1}{3}\sum_j\zeta^j(\mathbf{x}). \tag{16}$$

For the three points $\mathbf{a},\mathbf{b},\mathbf{c}$, we get the results given by Table 3.

We can observe that the inequality (14) is always satisfied, but equality is not.

Proposition 3.2 *The operator*

$$\mathcal{C}^{box}_{\sigma,\alpha}\left([\mathbf{z}](1:m),[\mathbf{x}]\right) = \mathcal{C}^{interval}_{\sigma,\alpha}\left([z_1](1:m),[x_1]\right)\times\cdots\times\mathcal{C}^{interval}_{\sigma,\alpha}\left([z_n](1:m),[x_n]\right)$$

is a contractor for the set $\mathbb{X}_\alpha$ defined by (5), in the case where the $\mathbb{Z}_j$ are boxes $[\mathbf{z}](j)$.

Table 3: Score values for the three points a, b, c.

	a	**b**	**c**
$\sigma(\zeta^1, \zeta^2, \zeta^3)$	$\frac{2}{3}$	0	$\frac{1}{3}$
$\sigma(\zeta_1^1, \zeta_1^2, \zeta_1^3)$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$
$\sigma(\zeta_2^1, \zeta_2^2, \zeta_2^3)$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$

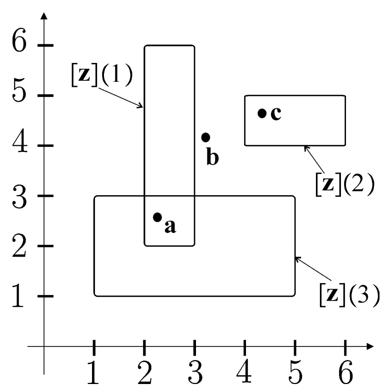


Figure 4: Only **a** belongs to at least two of the three boxes.

Proof: We have to check that $\mathcal{C}_{\sigma,\alpha}^{\text{box}}([\mathbf{x}])$ satisfies the properties of (13). Since all components $[x_i]$ of $[\mathbf{x}]$ are contracted by the contractor $\mathcal{C}_{\sigma,\alpha}^{\text{interval}}([z_i](1:m), [x_i])$ we have the contractance property. Let us show the consistency:

$$\begin{aligned}
\mathbb{X}_\alpha &= \left\{\mathbf{x} \in [\mathbf{x}] \mid \sigma\left(\zeta^1(\mathbf{x}), \ldots, \zeta^m(\mathbf{x})\right) \geq \alpha\right\} && \text{(see (5))} \\
&\subset \left\{\mathbf{x} \in [\mathbf{x}] \mid \forall i, \sigma\left(\zeta_i^1(x_i), \ldots, \zeta_i^m(x_i)\right) \geq \alpha\right\} && \text{(see (14))} \\
&= \textstyle\bigcap_i \left\{\mathbf{x} \in [\mathbf{x}] \mid \sigma\left(\zeta_i^1(x_i), \ldots, \zeta_i^m(x_i)\right) \geq \alpha\right\} \\
&= \left\{x_1 \in [x_1] \mid \sigma\left(\zeta_1^1(x_1), \ldots, \zeta_1^m(x_1)\right) \geq \alpha\right\} \times \\
&\quad \cdots \times \left\{x_n \in [x_n] \mid \sigma\left(\zeta_n^1(x_n), \ldots, \zeta_n^m(x_n)\right) \geq \alpha\right\}.
\end{aligned}$$

The monotonicity is a consequence of the fact that the Cartesian product is a monotonic operator. □

A contractor for $\mathbb{X}_\alpha$ is thus given by the following algorithm.

Algorithm $\mathcal{C}_{\sigma,\alpha}^{\text{box}}$(in: $[\mathbf{z}](1:m)$; inout: $[\mathbf{x}]$)
for all $i \in \{1, \ldots, n\}$
| $[x_i] = \mathcal{C}_{\sigma,\alpha}^{\text{interval}}([z_i](1:m), [x_i])$
return $[x_1] \times \cdots \times [x_n]$.

Algorithm 2. Contractor for the constraint $\sigma\left(\zeta^1(\mathbf{x}), \ldots, \zeta^m(\mathbf{x})\right) \geq \alpha$, when the $[z](i)$ are boxes.

Step 1 is a loop which allows a decomposition with respect to each axis of $\mathbf{x}$. *Step 2* calls the contractor $\mathcal{C}^{\text{interval}}_{\sigma,\alpha}$ that has been developed in Subsection 3.2.1 for the scalar case. *Step 3* returns the Cartesian product of all contracted intervals.

3.2.3 General Case

Proposition 3.3 *Assume that we have contractors $\mathcal{C}_{\mathbb{Z}_j}$, for the $\mathbb{Z}_j$. The operator*

$$\mathcal{C}^{set}_{\sigma,\alpha}\left(\mathbb{Z}_{1:m},[\mathbf{x}]\right)=\mathcal{C}^{box}_{\sigma,\alpha}\left(\left(\mathcal{C}_{\mathbb{Z}_1}([\mathbf{x}]),\dots,\mathcal{C}_{\mathbb{Z}_m}([\mathbf{x}])\right),[\mathbf{x}]\right)$$

is a contractor for the set $\mathbb{X}_\alpha$ defined by (5), in the case where the $\mathbb{Z}_j$ are any subsets of $\mathbb{R}^n$.

Proof: We have to check that $\mathcal{C}^{\text{set}}_{\sigma,\alpha}([\mathbf{x}])$ satisfies the properties of (13). The contractance property is trivial

$$\begin{aligned}\mathcal{C}^{\text{set}}_{\sigma,\alpha}\left(\mathbb{Z}_{1:m},[\mathbf{x}]\right) &\subset \mathcal{C}^{\text{box}}_{\sigma,\alpha}\left(\left([\mathbf{x}],\dots,[\mathbf{x}]\right)\right),[\mathbf{x}]) \\ &= \mathcal{C}^{\text{interval}}_{\sigma,\alpha}\left(\left([x_1],\dots,[x_1]\right)\right),[x_1]) \times \dots \\ &\quad \times \mathcal{C}^{\text{interval}}_{\sigma,\alpha}\left(\left([x_n],\dots,[x_n]\right)\right),[x_n]) \\ &= [x_1]\times\dots\times[x_n]=[\mathbf{x}]\end{aligned}$$

Let us show the consistency:

$$\begin{aligned}\mathbb{X}_\alpha &= \left\{\mathbf{x}\in[\mathbf{x}] \mid \sigma\left(\zeta^1(\mathbf{x}),\dots,\zeta^m(\mathbf{x})\right)\geq\alpha\right\} \text{ (see (5))} \\ &= \left\{\mathbf{x}\in[\mathbf{x}] \mid \sigma\left(\chi_{\mathbb{Z}_1}(\mathbf{x}),\dots,\chi_{\mathbb{Z}_m}(\mathbf{x})\right)\geq\alpha\right\} \\ &= \left\{\mathbf{x}\in[\mathbf{x}] \mid \sigma\left(\chi_{\mathbb{Z}_1\cap[\mathbf{x}]}(\mathbf{x}),\dots,\chi_{\mathbb{Z}_m\cap[\mathbf{x}]}(\mathbf{x})\right)\geq\alpha\right\} \\ &\subset \left\{\mathbf{x}\in[\mathbf{x}] \mid \sigma\left(\chi_{\mathcal{C}_{\mathbb{Z}_1}([\mathbf{x}])}(\mathbf{x}),\dots,\chi_{\mathcal{C}_{\mathbb{Z}_m}([\mathbf{x}])}(\mathbf{x})\right)\geq\alpha\right\} \\ &\subset \mathcal{C}^{\text{box}}_{\sigma,\alpha}\left(\left(\mathcal{C}_{\mathbb{Z}_1}([\mathbf{x}]),\dots,\mathcal{C}_{\mathbb{Z}_m}([\mathbf{x}])\right),[\mathbf{x}]\right).\end{aligned}$$

Finally, the monotonicity property directly comes from the monotonicity of the contractors $\mathcal{C}^{\text{set}}_{\sigma,\alpha}$ and the $\mathcal{C}_{\mathbb{Z}_j}$. □

A contractor $\mathcal{C}^{\text{set}}_{\sigma,\alpha}$ for $\mathbb{X}_\alpha$ is given by the following algorithm.

	Algorithm $\mathcal{C}^{\text{set}}_{\sigma,\alpha}$(in: $\mathcal{C}_{\mathbb{Z}_1},\dots,\mathcal{C}_{\mathbb{Z}_m}$; inout: $[\mathbf{x}]$)
1	for $j\in\{1,\dots,m\}$ do $[\mathbf{z}](j)=\mathcal{C}_{\mathbb{Z}_j}([\mathbf{x}])$
2	| $[\mathbf{x}]=\mathcal{C}^{\text{box}}_{\sigma,\alpha}([\mathbf{z}](1:m),[\mathbf{x}])$.

Algorithm 3. Contractor for the constraint $\sigma\left(\zeta^1(\mathbf{x}),\dots,\zeta^m(\mathbf{x})\right)\geq\alpha$, when the $\mathbb{Z}_j$ are general sets.

3.3 Characterization of α-cuts

Consider the α-cut $\mathbb{X}_\alpha$ (5) where $\zeta^j(\mathbf{x})=\chi_{\mathbb{Z}_j}(\mathbf{x})$ are the characteristic functions for the sets $\mathbb{Z}_j$. We want to compute an inner and an outer approximation for $\mathbb{X}_\alpha$. Assume that we have contractors $\mathcal{C}_{\mathbb{Z}_j}$, $\mathcal{C}_{\overline{\mathbb{Z}}_j}$ for the $\mathbb{Z}_j$, $\overline{\mathbb{Z}}_j$. A contractor for $\mathcal{C}_{\mathbb{X}_\alpha}$ and $\mathcal{C}_{\overline{\mathbb{X}}_\alpha}$ are given by

$$\begin{array}{rcl}\mathcal{C}_{\mathbb{X}_\alpha}([\mathbf{x}]) &=& \mathcal{C}^{\text{set}}_{\sigma,\alpha}(\mathcal{C}_{\mathbb{Z}_1},\dots,\mathcal{C}_{\mathbb{Z}_m},[\mathbf{x}]) \\ \mathcal{C}_{\overline{\mathbb{X}}_\alpha}([\mathbf{x}]) &=& \mathcal{C}^{\text{set}}_{\bar{\sigma},\bar{\alpha}}(\mathcal{C}_{\overline{\mathbb{Z}}_1},\dots,\mathcal{C}_{\overline{\mathbb{Z}}_m},[\mathbf{x}])\end{array} \tag{17}$$

where (see Proposition 2.1)

$$\begin{array}{rcl} \bar{\sigma}(a_1,\dots,a_m) & = & 1-\sigma(1-a_1,\dots,1-a_m) \\ \bar{\alpha} & = & 1-\alpha. \end{array} \tag{18}$$

Using a set inversion algorithm such as Sivia, inner and outer approximations of $\mathbb{X}_\alpha$ can thus be obtained. Recall that the principle of Sivia is to take a paving of boxes covering the set of all $\mathbf{x} \in \mathbb{R}^n$ of interest. We contract all boxes $[\mathbf{x}]$ using $\mathcal{C}_{\mathbb{X}_\alpha}$ and $\mathcal{C}_{\overline{\mathbb{X}}_\alpha}$. All part of $[\mathbf{x}]$ contracted by $\mathcal{C}_{\mathbb{X}_\alpha}$ are classified as outside $\mathbb{X}_\alpha$. All part of $[\mathbf{x}]$ contracted by $\mathcal{C}_{\overline{\mathbb{X}}_\alpha}$ are classified as inside $\mathbb{X}_\alpha$. If we apply the Sivia algorithm for different α-cuts, we obtain a paving similar to the one illustrated in Figure 1.

4 Application to Localization

An application for the proposed formalism is now presented. In mobile robotics, localization is an essential estimation problem where different granules of knowledge should be combined. The definition of the score function allows to finely choose the influence of each granule and obtain a result in the form of a fuzzy set. Thus, a robot is localized with a complex representation that takes into account the influence of each granule.

4.1 Underwater Localization With Sonar

Consider an underwater robot moving in a pool equipped with a directorial sonar, a compass and a manometer. We assume that the robot is static and that the pool is made with $w_{\max}$ segments $\mathbb{W}(w)$, $w \in \{1,\dots,w_{\max}\}$. Since the manometer provides us the depth and the compass returns the heading, the localization problem amounts to finding a point $\mathbf{x} = (x_1, x_2)^{\mathrm{T}}$ inside a horizontal plane from the distances returned by the sonar. The directorial sonar rotates and emits $k_{\max}$ ultrasonic sounds (or *pings*) toward different directions $\theta_1,\dots,\theta_{k_{\max}}$. For the kth ping, several echoes are returned. Each of them can be represented by a distance interval $[d_{k,\ell}]$. Only one is significant for us: the echo ℓ which corresponds to the nearest wall $\mathbb{W}(w)$ reached by the sound emitted in the direction θ_k. The triplet (k,w,ℓ) is then called an *inlier*. For non significant echoes, (k,w,ℓ) is called an *outlier*. Figure 5 shows a typical echo signal that could have been collected just after the kth ping. Note that since the distance d and the time t are linked by the relation $d = ct$, where $c = 1500$ m/s corresponds to the speed of the sound in seawater, the signal is represented with respect to d. The first echo $[d_{k,1}]$ may correspond to an echo from the surface of the water or any unmapped object. The second echo is the inlier (it corresponds to the echo returned by the nearest wall). The corresponding interval $[d_{k,2}]$ contains the true distance d_k. The last echo $[d_{k,3}]$ may correspond to a multiple echo or to a noise emitted by another robot. The first and the third echoes correspond to outliers.

Define the set

$$\mathbb{Z}_{k,w,\ell} = \{\mathbf{x} \in \mathbb{R}^2 \mid \begin{array}{l} \exists d \in [d_{k,\ell}], \\ \exists w \in \{1,\dots,w_{\max}\}, \\ \exists \mathbf{m} = (m_1,m_2) \in \mathbb{W}(w), \\ m_1 = x_1 + d\cos\theta_k, \\ m_2 = x_2 + d\sin\theta_k \quad \} \end{array} \tag{19}$$

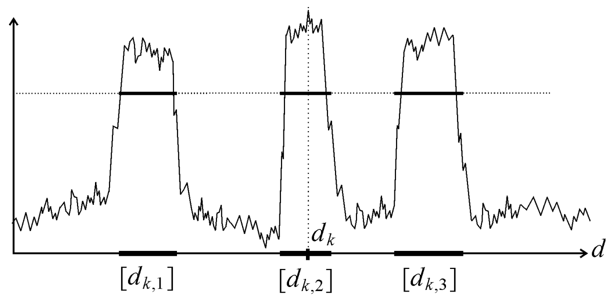


Figure 5: The signal collected by the sonar just after the ping emission.

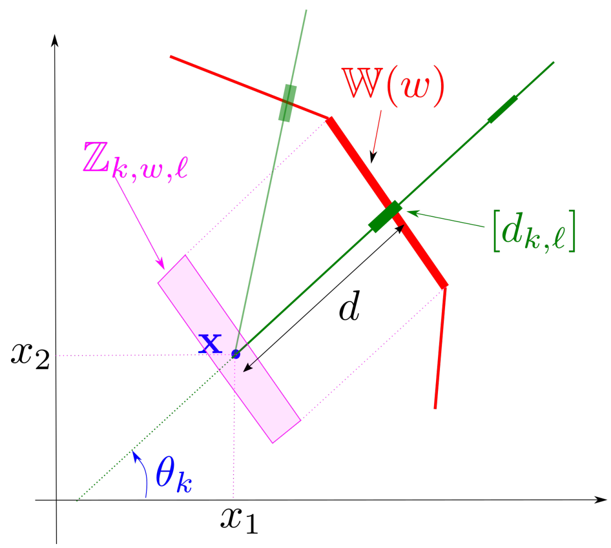


Figure 6: $\mathbb{Z}_{k,w,\ell}$ is the set of all positions for the robot consistent with the fact that the ℓth echo of the k ping corresponds to the wth wall $\mathbb{W}(w)$.

as illustrated by Figure 6. Define $\mathbb{Z}_0$ the set corresponding to all $\mathbf{x}$ that are in the pool. The set $\mathbb{Z}_{k,w,\ell}$ should contain the position $\mathbf{x}$ of the robot if (k, w, ℓ) corresponds to an inlier.

An optimal contractor can easily be built for $\mathbb{Z}_{k,w,\ell}$ [17]. Define for fuzzy set $\mathbb{X}$ the membership function

$$\mu_{\mathbb{X}}(\mathbf{x}) = \frac{1}{k_{\max}} \sum_{k\in\{1,\dots,k_{\max}\}} \max_{\ell\in\{1,\dots,\ell_{\max}(k)\}} \max_{w\in\{1,\dots,\bar{w}\}} \min\left(\zeta^{k,w,\ell}(\mathbf{x}), \zeta^0(\mathbf{x})\right) \quad (20)$$

where $\zeta^{k,w,\ell}$ and ζ^0 are the characteristic functions for $\mathbb{Z}_{k,w,\ell}$ and $\mathbb{Z}_0$, respectively. Note that we have the following equivalence

$$\begin{aligned} &\max_{\ell\in\{1,\dots,\ell_{\max}(k)\}} \max_{w\in\{1,\dots,w_{\max}\}} \min\left(\zeta^{k,w,\ell}(\mathbf{x}), \zeta^0(\mathbf{x})\right) = 1 \\ &\Leftrightarrow \exists \ell \in \{1,\dots,\ell_{\max}(k)\}, \exists w \in \{1,\dots,w_{\max}\}, \mathbf{x} \in \mathbb{Z}_{k,w,\ell} \cap \mathbb{Z}_0, \end{aligned} \quad (21)$$

i.e., there exists (ℓ, w) such that the ℓ^{th} echo and the wth wall $\mathbb{W}(w)$ are consistent with the position $\mathbf{x}$ of the robot and the fact that $\mathbf{x}$ is in the pool. Note also that $\mu_{\mathbb{X}}(\mathbf{x}) \geq \alpha$ when at least $k_{\max} \cdot \alpha$ pings are consistent.

4.2 Experimentation

Our approach has been tested with a real robot in a pool. The sonar used is a Tritech Micron (mechanical scanning sonar), fixed on a BlueROV2 of BlueRobotics. This sonar has a vertical beamwidth of 35° and was set for a range of 6 meters. The IMU (Inertial Measurement Unit) integrated into the flight controller (Pixhawk) returns the heading. The pool has the shape of a rectangle of 3 x 4 meters. A picture of the experimentation is shown in Figure 7.

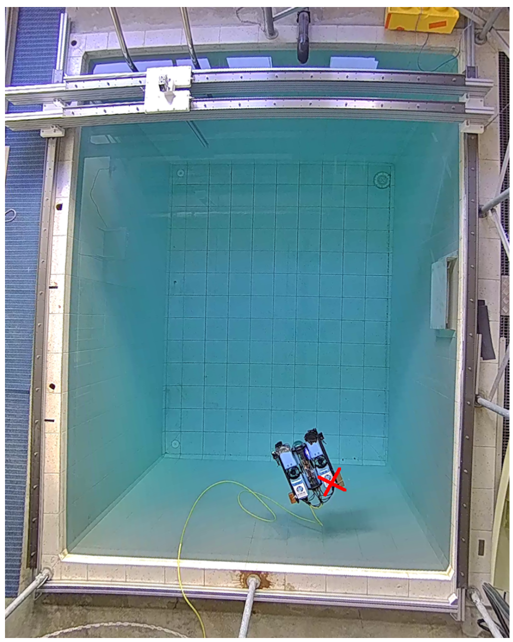


Figure 7: Picture of the BlueRobotics ROV in the pool localizing itself. The robot floats on the surface of the water, and the sonar is placed under the robot at the red cross position.

Data received from the sonar are presented in Figure 8. A peak detection algorithm is used to generate the interval data. In this experiment, the first peak received by the sonar often corresponds to the surface of the water, as the robot was barely submerged. Thus, the score function has been adapted to give more importance to the first echo and less to the second one. The score is drastically reduced for echoes that are not directed perpendicularly to the walls because it has been observed that such echoes are often less relevant (fewer chances to have an echo corresponding to a wall).

We propose to take as a set-membership estimator the $\hat{\alpha}$-cut $\mathbb{X}_{\hat{\alpha}}$ of the fuzzy set $\mathbb{X}$ where

$$\hat{\alpha} = \max \{\alpha \mid \mathbb{X}_\alpha \neq \emptyset\}. \tag{22}$$

The corresponding estimator can be interpreted as a generalization of the OMNE (Outlier Minimal Number Estimator) estimator presented in [33]. In OMNE, the score function σ is a simple sum.

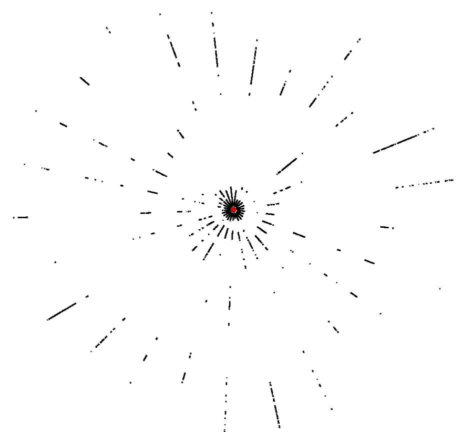


Figure 8: Representation of the data collected from the sonar. The red dot corresponds to the position of the robot.

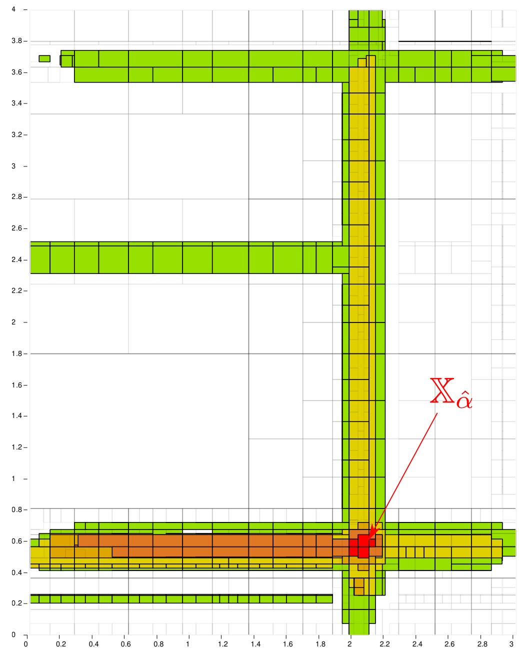
Our estimator generates a set $\mathbb{X}_{\hat{\alpha}}$ which contains the true position of the robot in the pool. As for most acoustic localization systems in a pool, many outliers exist. This is why we observe a small maximal score: $\hat{\alpha} = 0.15$. Figure 9 depicts the set $\mathbb{X}$. When the robot is moving, the procedure is called each time the sonar makes a turn. The localization algorithm takes less than 0.1 second on a standard laptop so it can be run online.

5 Conclusion

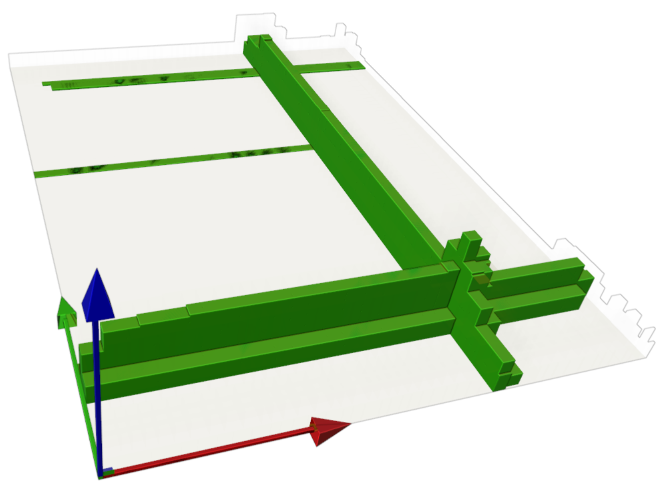
In this paper, we have proposed a new interval approach to deal with estimation problems. Our formulation is close to the priority approach to soft constraints proposed in [10]. The main idea is that when we cannot satisfy all the constraints, we should at least satisfy as many as possible, and a natural approach is to prioritize these constraints, from the absolutely required to the less required. This can be done through the set-membership function used in fuzzy set theory.

The formulation we have proposed requires the computation of α-cuts of a fuzzy set which can be done efficiently using interval approaches, at least when the membership function is piecewise constant. Our contributions are the following.

- We have proposed a formulation for combining contractors, which is more general than that proposed previously [9, 12, 47], since it allows the different weights in the constraints.
- Using the formulation based on the set-membership function, instead of the combination of contractors, we obtained an algorithm that is easy to implement.
- The fuzzy approach we propose for the resolution of localization problems has never been done before to our knowledge and is reliable even if we have partial



(a) Top view with α going from green to red



(b) 3D view

Figure 9: Representation of the fuzzy set $\mathbb{X}$. (a): On the x_1, x_2-plane. The plateau corresponding to $\mathbb{X}_{\hat{\alpha}}$ is painted red. It contains the true position of the robot. (b): 3D view of $\mathbb{X}$ in the (x_1, x_2, α)-space.

knowledge of the uncertainties. In this context, the fuzzy approach is more adapted than the classical Bayesian approach due to the difficulty we have in extracting significant data.

- We proposed an estimator which is more general than OMNE (Outlier Minimal Number Estimator) proposed in [33], since it allows us to deal with more complex situations where outlier occurrences are interdependent.

Our method has been validated on a real underwater robot for its localization using sonar data. We have shown good robustness to outliers and localization with a high degree of integrity. The fuzzy approach brought us the possibility to have a fine-tuning of the influence of each sonar data with respect to the degree of confidence we have in the measurements.

Acknowledgements

This work was supported by the French Government Defense Procurement and Technology Agency (DGA).

References

[1] H.H. Afshari, S.A. Gadsden, and S. Habibi. Gaussian filters for parameter and state estimation: A general review of theory and recent trends. *Signal Processing*, 135:218–238, 2017.

[2] G. Alefeld and J. Herzberger. *Introduction to Interval Computation*. Academic Press, New York, 1983.

[3] Ignacio Araya and Victor Reyes. Interval branch-and-bound algorithms for optimization and constraint satisfaction: a survey and prospects. *Journal of Global Optimization*, 65(4):837–866, 2016.

[4] F. Le Bars, J. Sliwka, O. Reynet, and L. Jaulin. State estimation with fleeting data. *Automatica*, 48(2):381–387, 2012.

[5] R. Boukezzoula, S. Galichet, and D. Coquin. From fuzzy regression to gradual regression: Interval-based analysis and extensions. *Information Sciences*, 441:18–40, 2018.

[6] R. Boukezzoula, S. Galichet, and L. Foulloy. Min and max operators for gradual intervals. *IEEE Trans. Fuzzy Syst.*, 26(6):3569–3578, 2018.

[7] R. Boukezzoula, L. Jaulin, and L. Foulloy. Thick gradual intervals: An alternative interpretation of type-2 fuzzy intervals and its potential use in type-2 fuzzy computations. *Engineering Applications of Artificial Intelligence*, 85:691–712, 2019.

[8] Q. Brefort, L. Jaulin, M. Ceberio, and V. Kreinovich. If we take into account that constraints are soft,then processing constraints becomes algorithmically solvable. In *Proceedings of the IEEE Series of Symposia on Computational Intelligence SSCI'2014*. Orlando, Florida, December 9-12, 2014.

[9] C. Carbonnel, G. Trombettoni, P. Vismara, and G. Chabert. Q-intersection algorithms for constraint-based robust parameter estimation. pages 2630–2636. AAAI Press, Québec City, Canada, 2014.

[10] M. Ceberio and V. Kreinovich. Towards an optimal approach to soft constraint problems. In *Proceedings of the SeventeenthWorld Congress Scientific Computation*. Applied Mathematics and Simulation, 2005.

[11] G. Chabert and L. Jaulin. A Priori Error Analysis with Intervals. *SIAM Journal on Scientific Computing*, 31(3):2214–2230, 2009.

[12] G. Chabert and L. Jaulin. Contractor Programming. *Artificial Intelligence*, 173:1079–1100, 2009.

[13] Bo Chen and Guoqiang Hu. Nonlinear state estimation under bounded noises. *Automatica*, 98:159–168, 2018.

[14] E. Colle and Galerne. Mobile robot localization by multiangulation using set inversion. *Robotics and Autonomous Systems*, 61(1):39–48, 2013.

[15] A.P. Dempster. Upper and lower probabilities induced by a multivalued mapping. *Annals of Mathematical Statistics*, 38(2):325–339, 1967.

[16] T. Denoeux. Constructing belief functions from sample data using multinomial confidence regions. *International Journal of Approximate Reasoning*, 42(3):228–252, 2006.

[17] B. Desrochers and L. Jaulin. A minimal contractor for the polar equation; application to robot localization. *Engineering Applications of Artificial Intelligence*, 55:83–92, 2016.

[18] M. Di Marco, A. Garulli, S. Lacroix, and A. Vicino. Set membership localization and mapping for autonomous navigation. *International Journal of Robust and Nonlinear Control*, 7(11):709–734, 2001.

[19] V. Drevelle and P. Bonnifait. High integrity gnss location zone characterization using interval analysis. In *ION GNSS*, 2009.

[20] D. Dubois, L. Jaulin, and H. Prade. *Thick Sets, Multiple-Valued Mappings and Possibility Theory*, pages 101–109. Springer, 2021.

[21] D. Dubois and H. Prade. *Fuzzy sets and systems: theory and applications*, volume 144. Academic Press, New York, NY, 1980.

[22] D. Dubois and H. Prade. *Possibility Theory*. Plenum, New York, 1988.

[23] D. Dubois and H. Prade. *Fundamentals of Fuzzy Sets*. Springer, 2000.

[24] Ronald Fagin. Combining fuzzy information: An overview. *SIGMOD Rec.*, 31(2):109–118, June 2002.

[25] E. Fogel and Y. F. Huang. On the value of information in system identification - bounded noise case. *Automatica*, 18(2):229–238, 1982.

[26] L. Jaulin, M. Kieffer, O. Didrit, and E. Walter. *Applied Interval Analysis, with Examples in Parameter and State Estimation, Robust Control and Robotics*. Springer-Verlag, London, 2001.

[27] L. Jaulin and E. Walter. Set inversion via interval analysis for nonlinear bounded-error estimation. *Automatica*, 29(4):1053–1064, 1993.

[28] L. Jaulin and E. Walter. Guaranteed robust nonlinear minimax estimation. *IEEE Transaction on Automatic Control*, 47(11):1857–1864, 2002.

[29] LeSheng Jin, Radko Mesiar, Surajit Borkotokey, and Martin Kalina. Certainty aggregation and the certainty fuzzy measures. *International Journal of Intelligent Systems*, 33(4):759–770, 2018.

[30] R. E. Kalman. Contributions to the theory of optimal control. *Bol. Soc. Mat. Mex.*, 5:102–119, 1960.

[31] M. Kieffer, L. Jaulin, E. Walter, and D. Meizel. Robust autonomous robot localization using interval analysis. *Reliable Computing*, 6(3):337–362, 2000.

[32] V. Kreinovich and S. Shary. Interval methods for data fitting under uncertainty: A probabilistic treatment. *Reliable Computing*, 23:105–140, 2016.

[33] H. Lahanier, E. Walter, and R. Gomeni. OMNE: a new robust membership-set estimator for the parameters of nonlinear models. *Journal of Pharmacokinetics and Biopharmaceutics*, 15:203–219, 1987.

[34] L. Longpré, C. Servin, and V. Kreinovich. Quantum computation techniques for gauging reliability of interval and fuzzy data. *International Journal of General Systems*, 40(1):99–109, 2011.

[35] R. E. Moore. *Methods and Applications of Interval Analysis*. SIAM, Philadelphia, PA, 1979.

[36] G. Nassreddine, F. Abdallah, and T. Denoeux. State estimation using interval analysis and belief function theory: Application to dynamic vehicle localization. *IEEE Transactions on Systems, Man, and Cybernetics*, 5(40):1205–1218, 2010.

[37] R. Neuland, M. Mantelli, B. Hummes, L. Jaulin, R. Maffei, E. Prestes, and M. Kolberg. Robust hybrid interval-probabilistic approach for the kidnapped robot problem. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 28(1), 2021.

[38] A. Neumaier. Clouds, fuzzy sets and probability intervals. *Reliable Computing*, 10(4):249–272, 2004.

[39] Bertrand Neveu, Martin De La Gorce, and Gilles Trombettoni. An interval branch and bound algorithm for parameter estimation. In *XIII GLOBAL OPTIMIZATION WORKSHOP GOW'16 4-8 September 2016*, volume 16, pages 133–136, 2016.

[40] Helene Piet-Lahanier and Eric Walter. Characterization of non-connected parameter uncertainty regions. *Mathematics and Computers in Simulation*, 32:553–560, 1990.

[41] O. Reynet, L. Jaulin, and G. Chabert. Robust tdoa passive location using interval analysis and contractor programming. In *Radar*, Bordeaux, France, 2009.

[42] S. Rohou, L. Jaulin, L. Mihaylova, F. Le Bars, and S. Veres. *Reliable robot localization*. ISTE Group, 2019.

[43] S. Rohou, L. Jaulin, M. Mihaylova, F. Le Bars, and S. Veres. Reliable non-linear state estimation involving time uncertainties. *Automatica*, pages 379–388, 2018.

[44] S.I. Roumeliotis and G.A. Bekey. Bayesian estimation and kalman filtering: a unified framework for mobile robot localization. In *Proceedings 2000 ICRA. Millennium Conference. IEEE International Conference on Robotics and Automation. Symposia Proceedings (Cat. No.00CH37065)*, volume 3, pages 2985–2992 vol.3, 2000.

[45] William Silvert. Symmetric summation: A class of operations on fuzzy sets. In Didier Dubois, Henri Prade, and Ronald R. Yager, editors, *Readings in Fuzzy Sets for Intelligent Systems*, pages 77–79. Morgan Kaufmann, 1993.

[46] S. Thrun, W. Bugard, and D. Fox. *Probabilistic Robotics*. MIT Press, Cambridge, M.A., 2005.

[47] G. Trombettoni and G. Chabert. Constructive Interval Disjunction. In *Proc. CP, Constraint Programming*, pages 635–650, LNCS 4741, 2007.

[48] E. Walter and L. Pronzato. *Identification of Parametric Models from Experimental Data*. Springer-Verlag, London, UK, 1997.

[49] L. Zadeh. Quantitative fuzzy semantics. *Information Sciences*, (3):159–176, 1971.