Interval Finite Elements and Uncertainty in Engineering Analysis and Design

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Outline

- Introduction
- Uncertainty
- Interval Finite Elements
- Element-By-Element
- Examples
- Conclusions

Acknowledgment

Robert Mullen

Hao Zhang

Outline

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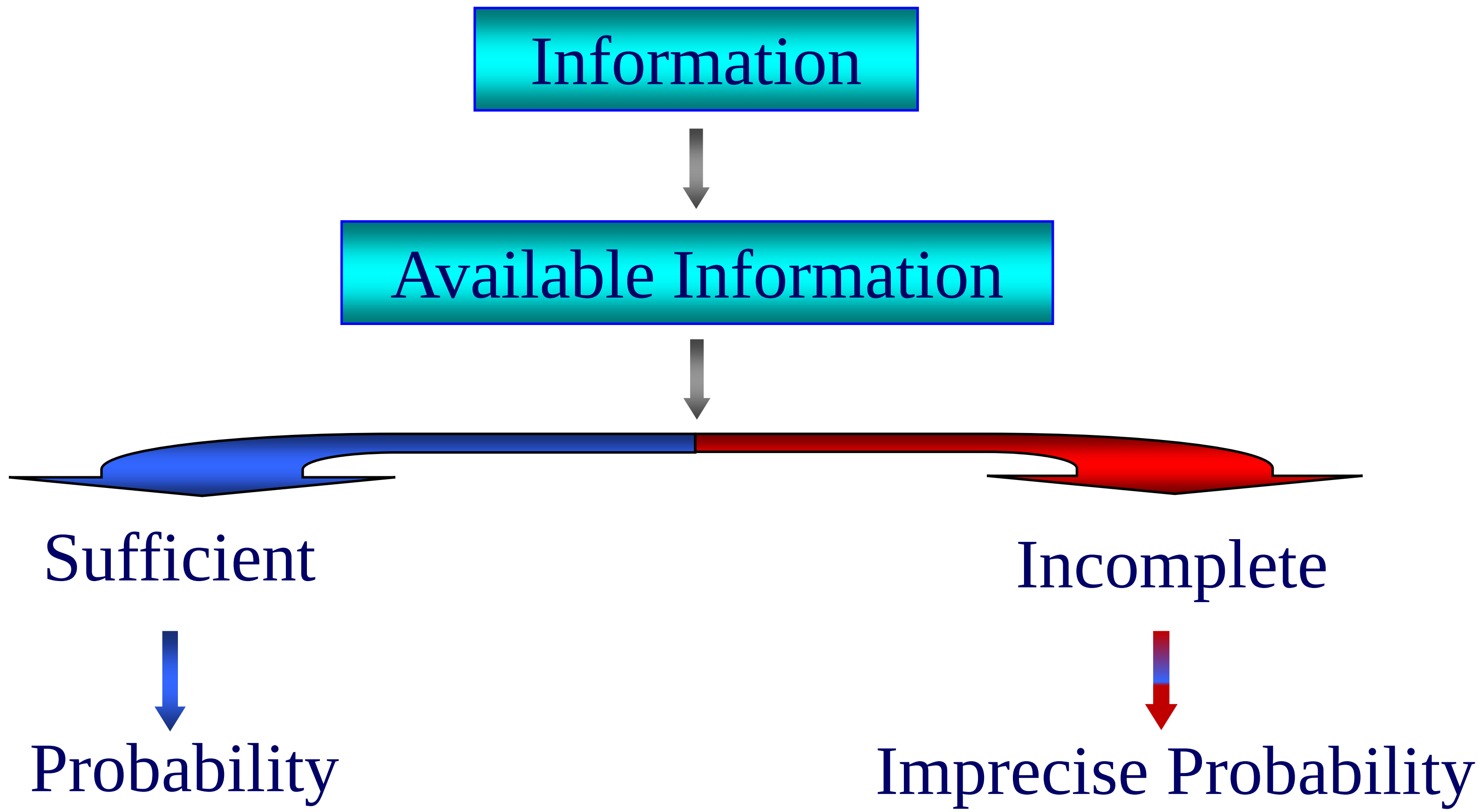
Introduction- Uncertainty

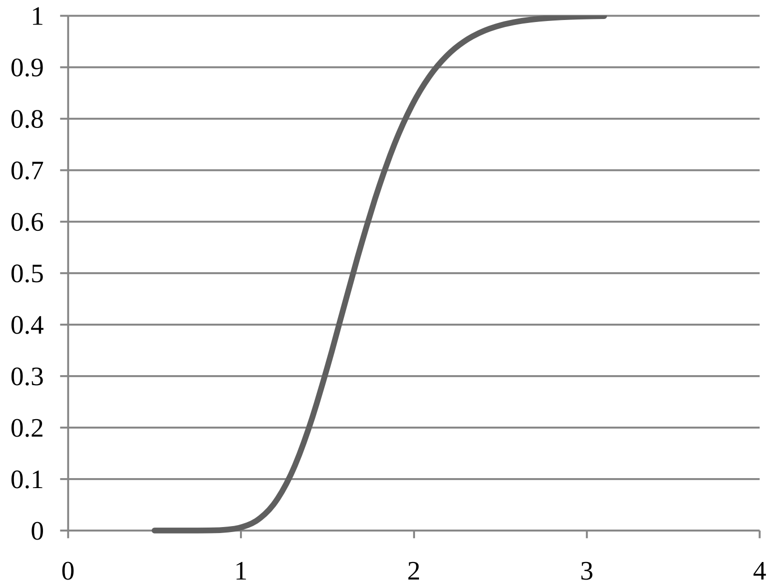
- Uncertainty is unavoidable in engineering system
  - Structural mechanics entails uncertainties in material, geometry and load parameters (aleatory-epistemic)
- Probabilistic approach is the traditional approach
  - Requires sufficient information to validate the probabilistic model
  - Credibility of probabilistic approach when data is insufficient (Elishakoff, 1995; Williamson, 1990, Ferson and Ginzburg, 1996; Möller and Beer, 2007)

Introduction- Uncertainty



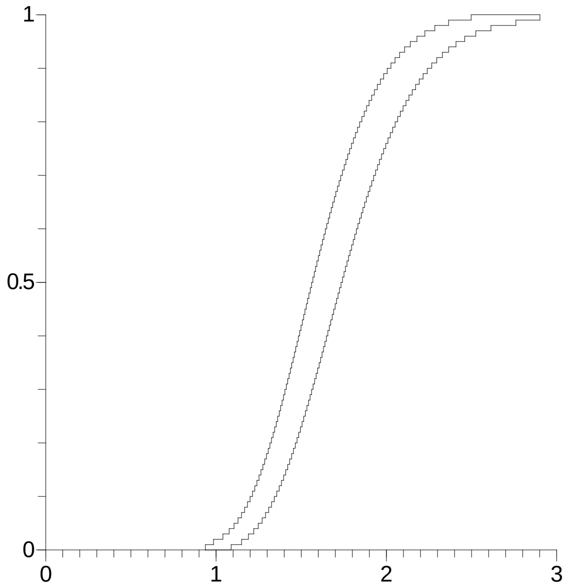
Introduction- Uncertainty

Lognormal



Probability

Lognormal with interval mean



Imprecise Probability

Tucker, W. T. and Ferson, S. , Probability bounds analysis in environmental risk assessments, Applied Biomathematics, 2003.

Introduction- Interval Approach

- Nonprobabilistic approach for uncertainty modeling when only range information (tolerance) is available

$$t = t_0 \pm \delta$$

- Represents an uncertain quantity by giving a range of possible values

$$t = [t_0 - \delta,\ t_0 + \delta]$$

- How to define bounds on the possible ranges of uncertainty?
  - experimental data, measurements, statistical analysis, expert knowledge

Introduction- Why Interval?

- Simple and elegant
- Conforms to practical tolerance concept
- Describes the uncertainty that can not be appropriately modeled by probabilistic approach
- Computational basis for other uncertainty approaches (e.g., fuzzy set, random set, imprecise probability)

- **Provides guaranteed enclosures**

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- Interval Arithmetic
- Interval Finite Elements
- Element-By-Element
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Interval arithmetic

- Interval number represents a range of possible values within a closed set

$$\boldsymbol{x} \equiv [\underline{x}, \overline{x}] := \{x \in R \mid \underline{x} \le x \le \overline{x}\}$$

Properties of Interval Arithmetic

Let x, y and z be interval numbers

1. Commutative Law

$$x + y = y + x$$

$$xy = yx$$

2. Associative Law

$$x + (y + z) = (x + y) + z$$

$$x(yz) = (xy)z$$

3. ***Distributive Law does not always hold,*** *but*

$$x(y + z) \subseteq xy + xz$$

Sharp Results – Overestimation

- The ***DEPENDENCY*** problem arises when one or several variables occur more than once in an interval expression

  - $f(\mathbf{x}) = \mathbf{x} - \mathbf{x}, \qquad \mathbf{x} = [1, 2]$
  - $f(\mathbf{x}) = [1 - 2, 2 - 1] = [-1, 1] \neq 0$
  - ~~$f(\mathbf{x}, \mathbf{y}) = \{ f(x, y) = x - y \mid x \in \mathbf{x}, y \in \mathbf{y} \}$~~
  - $f(\mathbf{x}) = \mathbf{x}(1 - 1) \quad \Rightarrow \quad f(\mathbf{x}) = 0$
  - $f(\mathbf{x}) = \{ f(x) = x - x \mid x \in \mathbf{x} \}$

Sharp Results – Overestimation

- Let ***a*, *b*, *c*** and ***d*** be independent variables, each with interval [1, 3]

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} \boldsymbol{a} & -\boldsymbol{b} \\ -\boldsymbol{c} & \boldsymbol{d} \end{pmatrix}, \quad A \times \boldsymbol{B} = \begin{pmatrix} [-2,2] & [-2,2] \\ [-2,2] & [-2,2] \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \boldsymbol{B}_{phys} = \begin{pmatrix} \boldsymbol{b} & -\boldsymbol{b} \\ -\boldsymbol{b} & \boldsymbol{b} \end{pmatrix}, \quad A \times \boldsymbol{B}_{phys} = \begin{pmatrix} [\boldsymbol{b}-\boldsymbol{b}] & [\boldsymbol{b}-\boldsymbol{b}] \\ [\boldsymbol{b}-\boldsymbol{b}] & [\boldsymbol{b}-\boldsymbol{b}] \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad B^{*}_{phys} = \boldsymbol{b} \times \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad A \times \boldsymbol{B}^{*}_{phys} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

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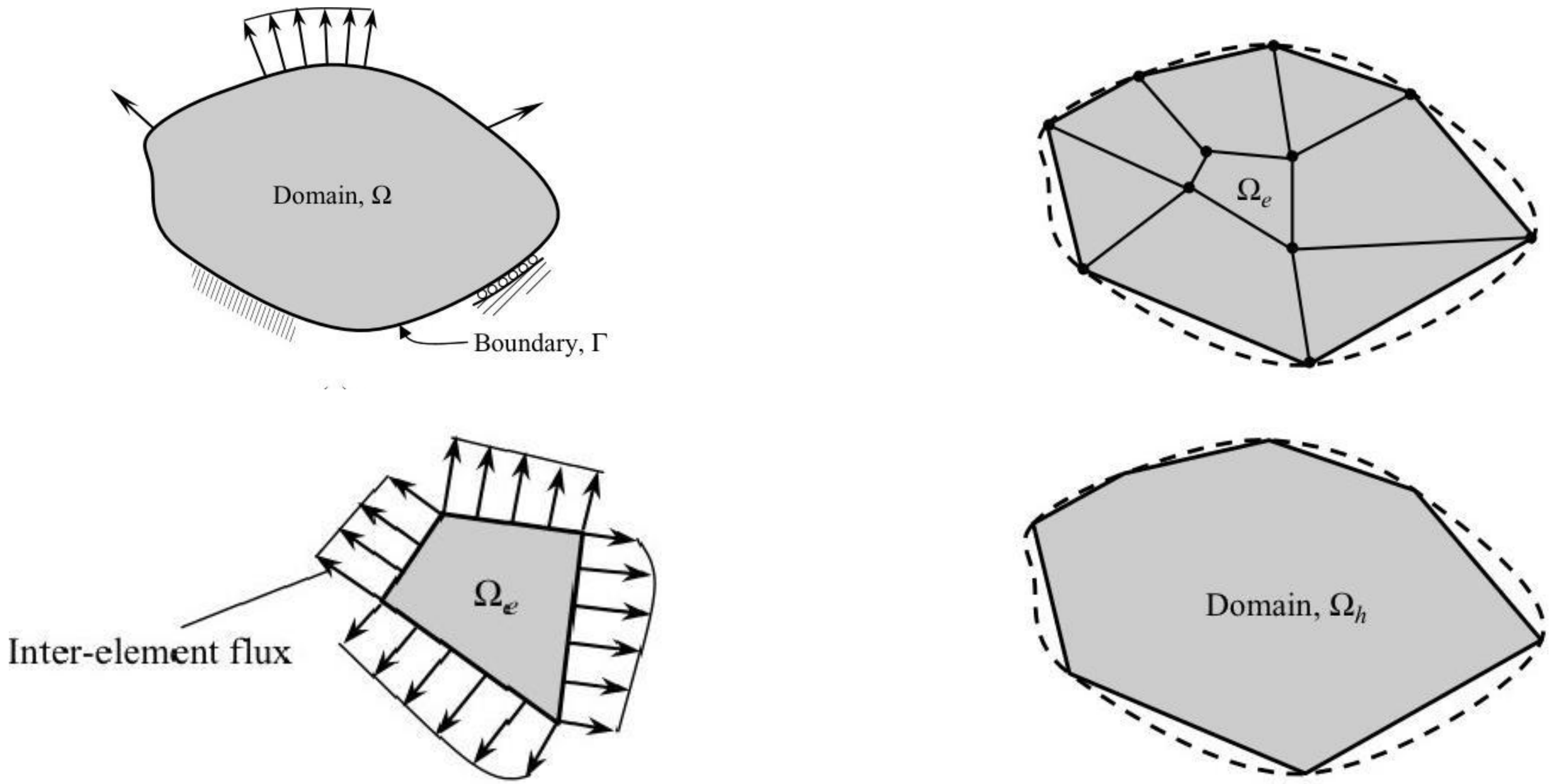
Finite Elements

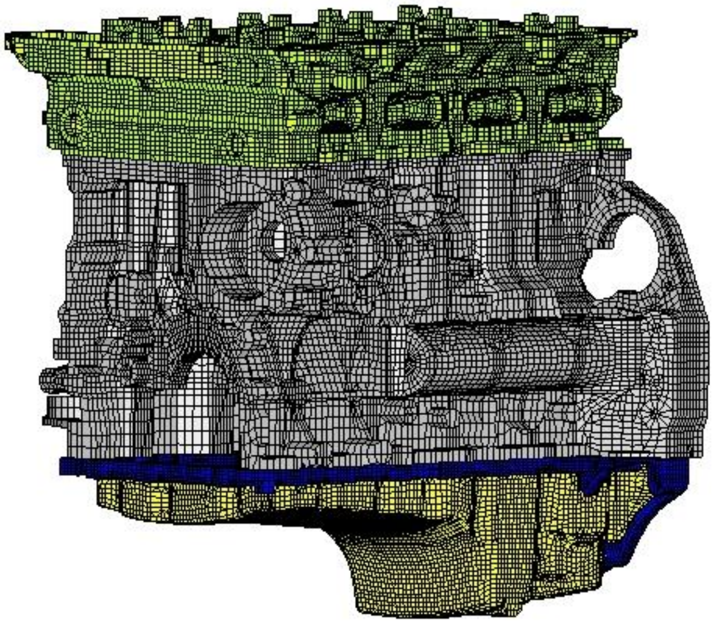
Finite Element Method (FEM) is a numerical method that provides approximate solutions to differential equations (ODE and PDE)

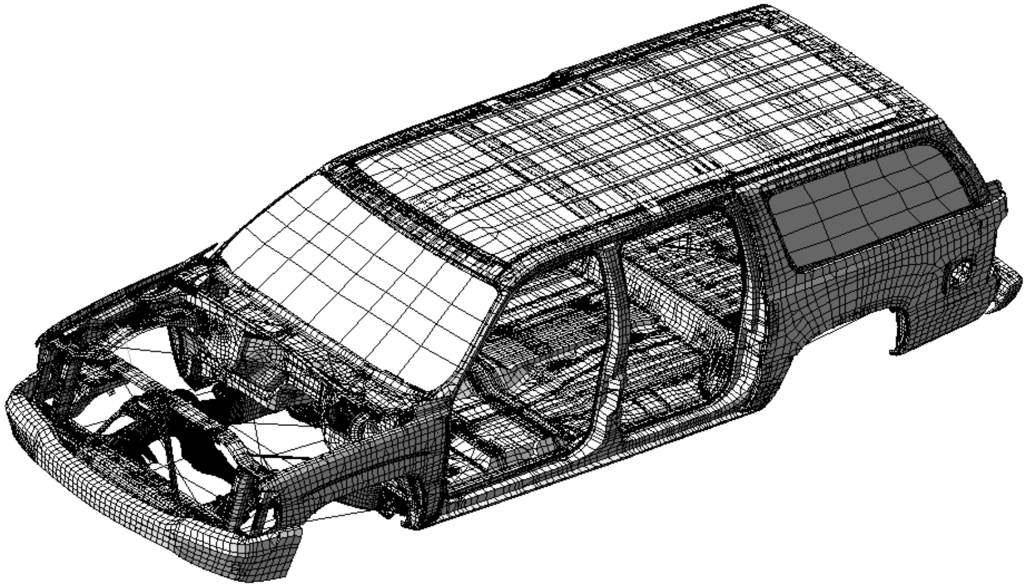
Finite Elements



Representation of a 2-D domain by a collection of triangles and quadrilaterals (Reddy, J. N. An intr. to the FEM, 3rd ed., 2006)

Finite Elements





Finite Element Model (courtesy of Prof. Mourelatous)

500,000-1,000,000 equations

Finite Elements

For example, in the case of one dimensional-second order differential equation

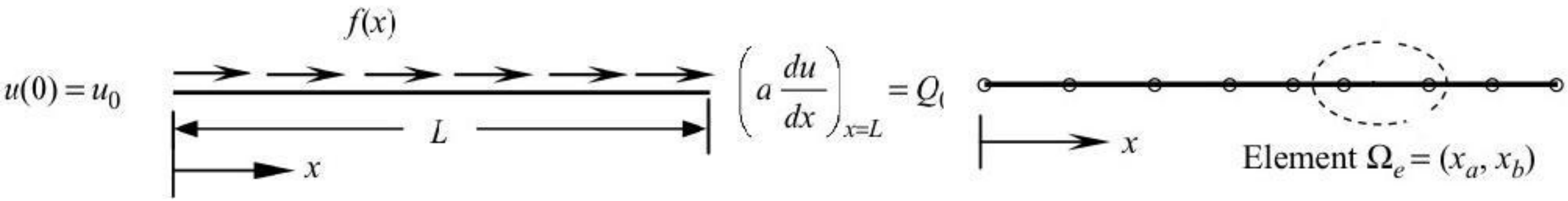
$$-\frac{d}{dx}(a(x)\frac{du}{dx}) + c(x)u = f(x) \qquad for \qquad 0 < x < L$$

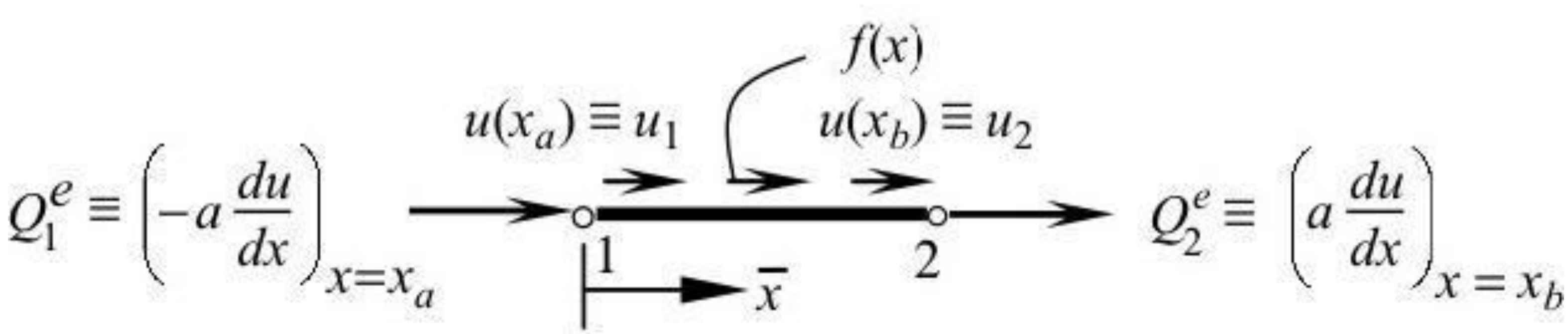
with the boundary conditions

$$u(0) = u_0, \quad (a\frac{du}{dx})\bigg|_{x=L} = Q_0$$

Finite Elements

The domain is descritized into finite elements





Finite Element

(Reddy, J. N. An intr. to the FEM, 3rd ed., 2006)

Finite Elements

The weak form over an element: find *u*(*x*) for all *w* in the appropriate Hilbert space

$$0 = \int_{x_a}^{x_b} w\left[-\frac{d}{dx}\left(a(x)\frac{du}{dx}\right) + c(x)u - f(x)\right] dx$$

$$= \int_{x_a}^{x_b}\left[a\frac{dw}{dx}\frac{du}{dx} + cwu - wf\right] dx - \left[w \cdot a\frac{du}{dx}\right]_{x_a}^{x_b}$$

$$= \int_{x_a}^{x_b}\left[a\frac{dw}{dx}\frac{du}{dx} + cwu - wf\right] dx - w(x_a)Q_{\mathrm{a}} - w(x_b)\cdot Q_b$$

Finite Elements

The finite element approximation

$$u(x) \approx u_h(x) = \sum_{j=1}^{n} u_j \psi_j(x)$$

and the finite element model is

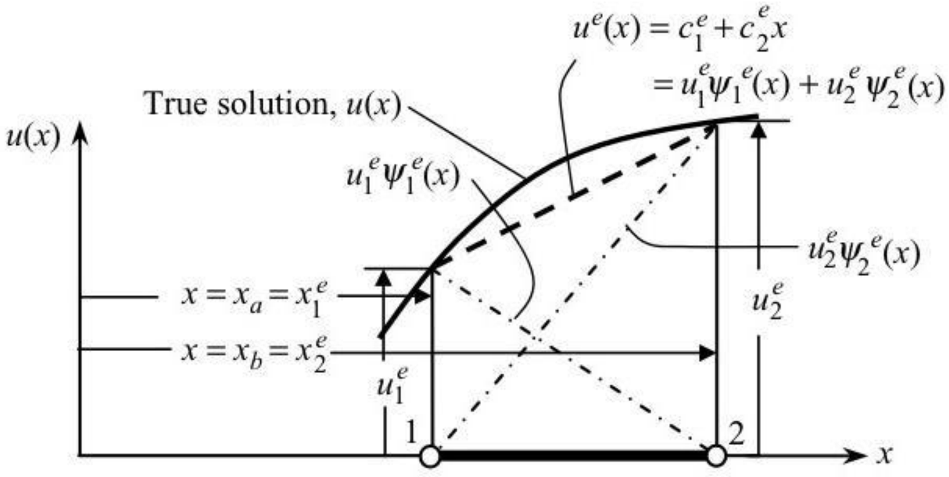
$$[K]\{u\} = \{F\}$$

$$K_{ij} = \int_{x_a}^{x_b} \left(a \frac{d\psi_i}{dx} \frac{d\psi_j}{dx} + c\psi_i \psi_j \right) dx,$$

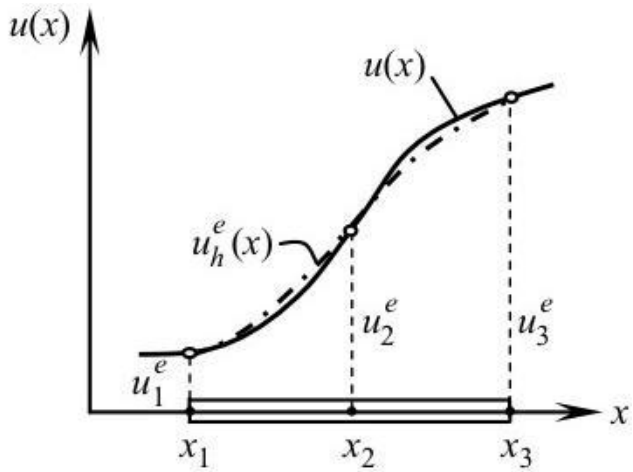
$$F_i = \int_{x_a}^{x_b} \psi_i f dx + \psi_i(x_a) Q_a + \psi_i(x_b) Q_b$$

Finite Elements

The finite element solution is of the form



Linear Element



Quadratic Element

(Reddy, J. N. An intr. to the FEM, 3rd ed., 2006)

Finite Elements- Uncertainty& Errors

- ❑ Mathematical model (validation)
- ❑ Discretization of the mathematical model into a computational framework (verification)
- ❑ Parameter uncertainty (loading, material properties)
- ❑ Rounding errors

Interval Finite Elements (IFEM)

- Follows conventional FEM
- Loads, geometry and material property are expressed as interval quantities
- System response is a function of the interval variables and therefore varies in an interval
- Computing the exact response range is proven NP-hard
- The problem is to estimate the bounds on the unknown exact response range based on the bounds of the parameters

FEM- Inner-Bound Methods

- Combinatorial method (Muhanna and Mullen 1995, Rao and Berke 1997)
- Sensitivity analysis method (Pownuk 2004)
- Perturbation (Mc William 2000)
- Monte Carlo sampling method
- Need for alternative methods that achieve
  - Rigorousness – guaranteed enclosure
  - Accuracy – sharp enclosure
  - Scalability – large scale problem
  - Efficiency

IFEM- Enclosure

- Linear static finite element
  - Muhanna, Mullen, 1995, 1999, 2001,and Zhang 2004
  - Popova 2003, and Kramer 2004
  - Neumaier and Pownuk 2004
  - Corliss, Foley, and Kearfott 2004
- Heat Conduction
  - Pereira and Muhanna 2004
- Dynamic
  - Dessombz, 2000
- Free vibration-Buckling
  - Modares, Mullen 2004, and Bellini and Muhanna 2005

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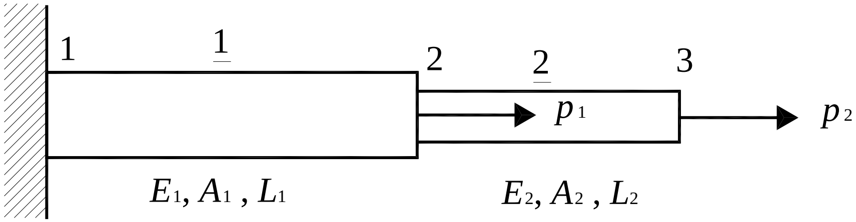
Naïve interval FEA



$$E_1A_1/L_1 = \boldsymbol{k}_1 = [0.95,\ 1.05],$$
$$E_2A_2/L_2 = \boldsymbol{k}_2 = [1.9,\ 2.1],$$
$$p_1 = 0.5, \quad p_2 = 1$$

$$\begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \Rightarrow \begin{pmatrix} [2.85,\ 3.15] & [-2.1,\ -1.9] \\ [-2.1,\ -1.9] & [1.9,\ 2.1] \end{pmatrix}\begin{pmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 1 \end{pmatrix}$$

- exact solution: $\boldsymbol{u}_2$ = [1.429, 1.579], $\boldsymbol{u}_3$ = [1.905, 2.105]
- naïve solution: $\boldsymbol{u}_2$ = [−0.052, 3.052], $\boldsymbol{u}_3$ = [0.098, 3.902]
- interval arithmetic assumes that all coefficients are independent
- uncertainty in the response is severely overestimated (1900%)

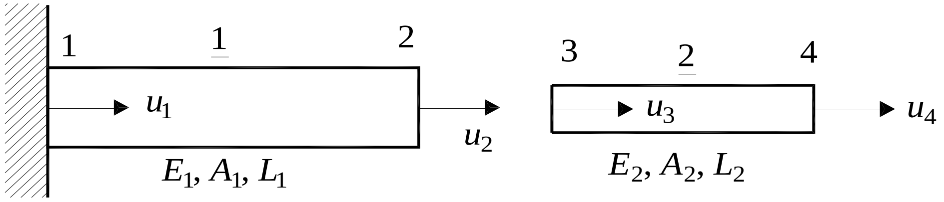
Element-By-Element

Element-By-Element (EBE) technique

- elements are detached – no element coupling
- structure stiffness matrix is block-diagonal (k_1 ,…, k_{Ne})
- the size of the system is increased

$$u = (u_1, \ldots, u_{Ne})^T$$

- need to impose necessary constraints for compatibility and equilibrium



Element-By-Element model

Element-By-Element

Suppose the modulus of elasticity is interval:

$$\boldsymbol{E} = \hat{E}(1+\boldsymbol{\delta})$$

$\boldsymbol{\delta}$: zero-midpoint interval

The element stiffness matrix can be split into two parts,

$$\boldsymbol{k} = \hat{k}(I+\boldsymbol{d}) = \hat{k} + \hat{k}\boldsymbol{d}$$

$\hat{k}$: deterministic part, element stiffness matrix evalued using $\hat{E}$,

$\hat{k}\boldsymbol{d}$: interval part

$\boldsymbol{d}$: interval diagonal matrix, diag($\boldsymbol{\delta}$,...,$\boldsymbol{\delta}$).

Element-By-Element

- Element stiffness matrix: $\boldsymbol{k} = \hat{k}(I + \boldsymbol{d})$

- Structure stiffness matrix:

$$\boldsymbol{K} = \hat{K}(I + \boldsymbol{D}) = \hat{K} + \hat{K}\boldsymbol{D}$$

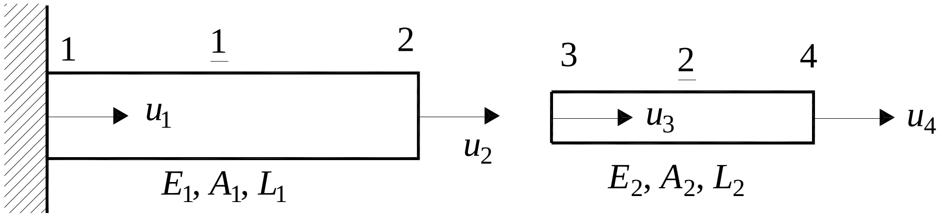
or

$$\boldsymbol{K} = \begin{pmatrix} \boldsymbol{k}_1 & & \\ & \ddots & \\ & & \boldsymbol{k}_{N_e} \end{pmatrix} = \begin{pmatrix} \hat{k}_1 & & \\ & \ddots & \\ & & \hat{k}_{N_e} \end{pmatrix} \left(I + \begin{pmatrix} \boldsymbol{d}_1 & & \\ & \ddots & \\ & & \boldsymbol{d}_{N_e} \end{pmatrix} \right)$$

Constraints

Impose necessary constraints for compatibility and equilibrium

- ❑ Penalty method
- ❑ Lagrange multiplier method



Element-By-Element model

Constraints – penalty method

Constraint conditions: $c\boldsymbol{u} = 0$

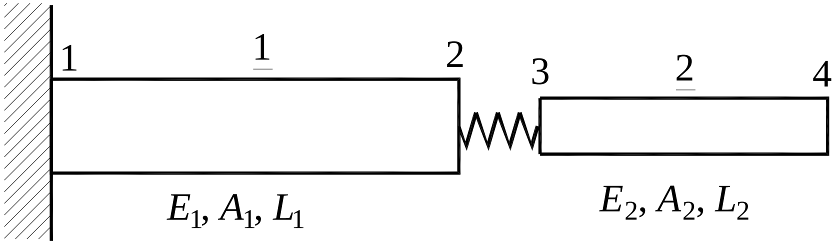
Using the penalty method:

$$(\boldsymbol{K} + Q)\boldsymbol{u} = \boldsymbol{p}$$

Q : penalty matrix, $Q = c^T \eta c$

η : diagonal matrix of penalty number η_i

Requires a careful choice of the penatly number



A spring of large stiffness is added to force node 2 and node 3 to have the same displacement.

Constraints – Lagrange multiplier

Constraint conditions: $c\boldsymbol{u} = 0$

Using the Lagrange multiplier method:

$$\begin{pmatrix} \boldsymbol{K} & c^T \\ c & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \boldsymbol{p} \\ 0 \end{pmatrix}$$

$\boldsymbol{\lambda}$: Lagrange multiplier vector, introdued as new unknowns

Load in EBE

Nodal load $\boldsymbol{p}_b$

$$\boldsymbol{p}_b = (\boldsymbol{p}_1, \ldots, \boldsymbol{p}_{Ne})$$

where $\boldsymbol{p}_i = \int \psi^T \boldsymbol{f}(x)\, dx$

Suppose the surface traction $\boldsymbol{f}(x)$ is described by an interval function: $\boldsymbol{f}(x) = \sum_{j=0}^{m} \boldsymbol{a}_j x^j$

$\boldsymbol{p}_b$ can be rewritten as

$$\boldsymbol{P}_b = M\,\boldsymbol{F}$$

M : deterministic matrix

$\boldsymbol{F}$: interval vector containing the interval coefficients of the surface traction

Fixed point iteration

- For the interval equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$,
  - preconditioning: $R\boldsymbol{A}\boldsymbol{x} = R\boldsymbol{b}$, R is the preconditioning matrix
  - transform it into $\boldsymbol{g}(\boldsymbol{x}^*) = \boldsymbol{x}^*$:

$$R\,\boldsymbol{b} - R\boldsymbol{A}\,x_0 + (I - R\boldsymbol{A})\,\boldsymbol{x}^* = \boldsymbol{x}^*, \qquad \boldsymbol{x} = \boldsymbol{x}^* + x_0$$

- **Theorem** (Rump, 1990): for some interval vector $\boldsymbol{x}^*$,

  if $\quad \boldsymbol{g}(\boldsymbol{x}^*) \subseteq \mathrm{int}\,(\boldsymbol{x}^*)$

  then $\quad \boldsymbol{A}^H \boldsymbol{b} \subseteq \boldsymbol{x}^* + x_0$

- Iteration algorithm:

  iterate: $\boldsymbol{x}^{*(l+1)} = \boldsymbol{z} + \boldsymbol{G}(\boldsymbol{\varepsilon} \cdot \boldsymbol{x}^{*(l)})$

  where $\boldsymbol{z} = R\boldsymbol{b} - R\boldsymbol{A}x_0,\ \boldsymbol{G} = I - R\boldsymbol{A},\ R = \hat{A}^{-1},\ \hat{A}x_0 = \hat{b}$

- No dependency handling

Fixed point iteration

- Interval FEA calls for a modified method which exploits the special form of the structure equations

$$(\boldsymbol{K}+Q)\boldsymbol{u}=\boldsymbol{p} \text{ with } \boldsymbol{K}=\hat{K}+\hat{K}\boldsymbol{D}$$

- Choose $R=(\hat{K}+Q)^{-1}$, construct iterations:

$$\begin{aligned}\boldsymbol{u}^{*(l+1)} &= R\boldsymbol{p}-R(\boldsymbol{K}+Q)u_0+(I-R(\boldsymbol{K}+Q))(\boldsymbol{\varepsilon}\cdot\boldsymbol{u}^{*(l)})\\ &= R\boldsymbol{p}-u_0-R\hat{K}\boldsymbol{D}(u_0+\boldsymbol{\varepsilon}\cdot\boldsymbol{u}^{*(l)})\\ &= R\boldsymbol{p}-u_0-R\hat{K}\boldsymbol{M}^{(l)}\boldsymbol{\Delta}\end{aligned}$$

if $\boldsymbol{u}^{*(l+1)}\subseteq \text{int}(\boldsymbol{u}^{*(l)})$, then $\boldsymbol{u}=\boldsymbol{u}^{*(l+1)}+u_0=R\boldsymbol{p}-R\check{K}\boldsymbol{M}^{(l)}\boldsymbol{\Delta}$

$\boldsymbol{\Delta}$: interval vector, $\boldsymbol{\Delta}=(\delta_1,\ldots,\delta_{N_e})^T$

The interval variables $\delta_1,\ldots,\delta_{N_e}$ appear only once in each iteration.

Convergence of fixed point

- The algorithm converges if and only if $\rho(|\boldsymbol{G}|) < 1$

  smaller $\rho(|\boldsymbol{G}|) \Rightarrow$ less iterations required,

  and less overestimation in results

- To minimize $\rho(|\boldsymbol{G}|)$:

  - choose $R = \hat{A}^{-1}$ so that $\boldsymbol{G} = \boldsymbol{I} - R\boldsymbol{A}$ has a small spectral radius

  - reduce the overestimation in $\boldsymbol{G}$

$$\boldsymbol{G} = I - R\boldsymbol{A} = \boldsymbol{I} - (\hat{K} + Q)^{-1}(\hat{K} + Q + \hat{K}\boldsymbol{D}) = -R\hat{K}\boldsymbol{D}$$

Stress calculation

- Conventional method:

$$\boldsymbol{\sigma} = \boldsymbol{C}B\boldsymbol{u}_e, \text{ (severe overestimation)}$$

C: elasticity matrix, *B*: strain-displacement matrix

- Present method:

$$\boldsymbol{E} = (1+\boldsymbol{\delta})\hat{E}, \quad \boldsymbol{C} = (1+\boldsymbol{\delta})\hat{C}$$

$$\begin{aligned}\boldsymbol{\sigma} &= \boldsymbol{C}BL\boldsymbol{u} \\ &= \boldsymbol{C}BL(R\boldsymbol{p} - R\hat{C}\boldsymbol{M}^{(l)}\boldsymbol{\Delta}) \\ &= (1+\boldsymbol{\delta})(\hat{C}BLR\boldsymbol{p} - \hat{C}BLR\hat{K}\boldsymbol{M}^{(l)}\boldsymbol{\Delta})\end{aligned}$$

L: Boolean matrix, $L\boldsymbol{u} = \boldsymbol{u}_e$

Element nodal force calculation

- Conventional method:

$$\boldsymbol{f} = T_e(\boldsymbol{k}\boldsymbol{u}_e - \boldsymbol{p}_e), \quad \text{(severe overestimation)}$$

- Present method:

in the EBE model, $$T(\boldsymbol{K}\boldsymbol{u} - \boldsymbol{p}_b) = \begin{pmatrix} (\boldsymbol{T}_e)_1(\boldsymbol{k}_1(\boldsymbol{u}_e)_1 - (\boldsymbol{p}_e)_1) \\ \vdots \\ (\boldsymbol{T}_e)_{N_e}(\boldsymbol{k}_{N_e}(\boldsymbol{u}_e)_{N_e} - (\boldsymbol{p}_e)_{N_e}) \end{pmatrix}$$

from $(\boldsymbol{K} + Q)\boldsymbol{u} = \boldsymbol{p}_c + \boldsymbol{p}_b \Rightarrow T(\boldsymbol{K}\boldsymbol{u} - \boldsymbol{p}_b) = T(\boldsymbol{p}_c - Q\boldsymbol{u})$

Calculate $T(\boldsymbol{p}_c - Q\boldsymbol{u})$ to obtain the element nodal forces for all elements.

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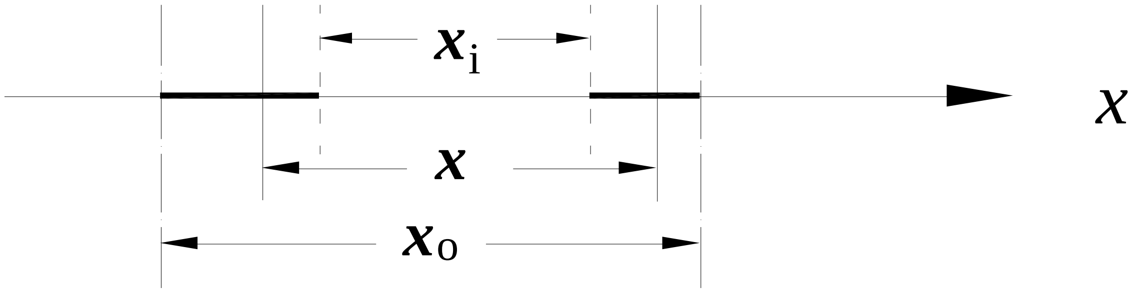
Numerical example

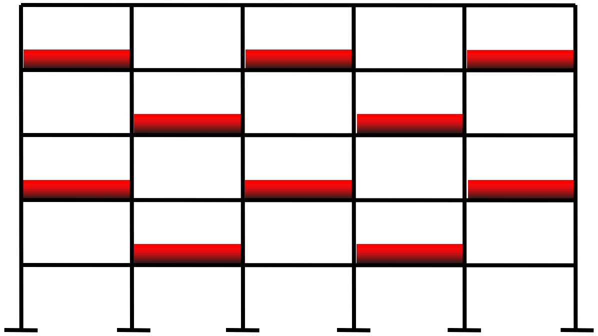
- Examine the rigorousness, accuracy, scalability, and efficiency of the present method
- Comparison with the alternative methods
  - the combinatorial method, sensitivity analysis method, and Monte Carlo sampling method
  - these alternative methods give inner estimation

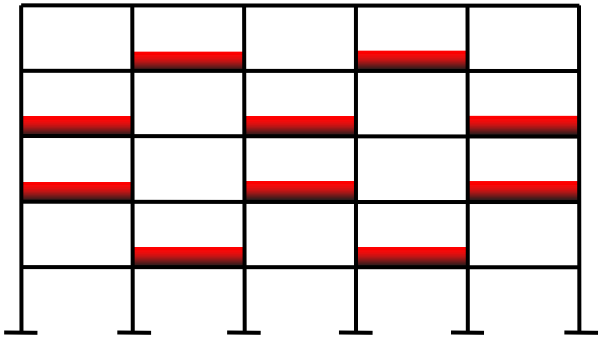


$\boldsymbol{x}$: exact solution, $\boldsymbol{x}_i$: inner bound, $\boldsymbol{x}_o$: outer bound

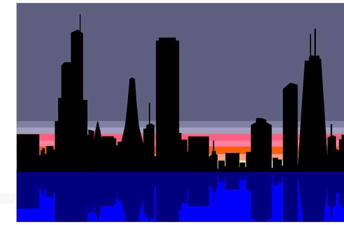
Examples – Load Uncertainty

- Four-bay forty-story frame

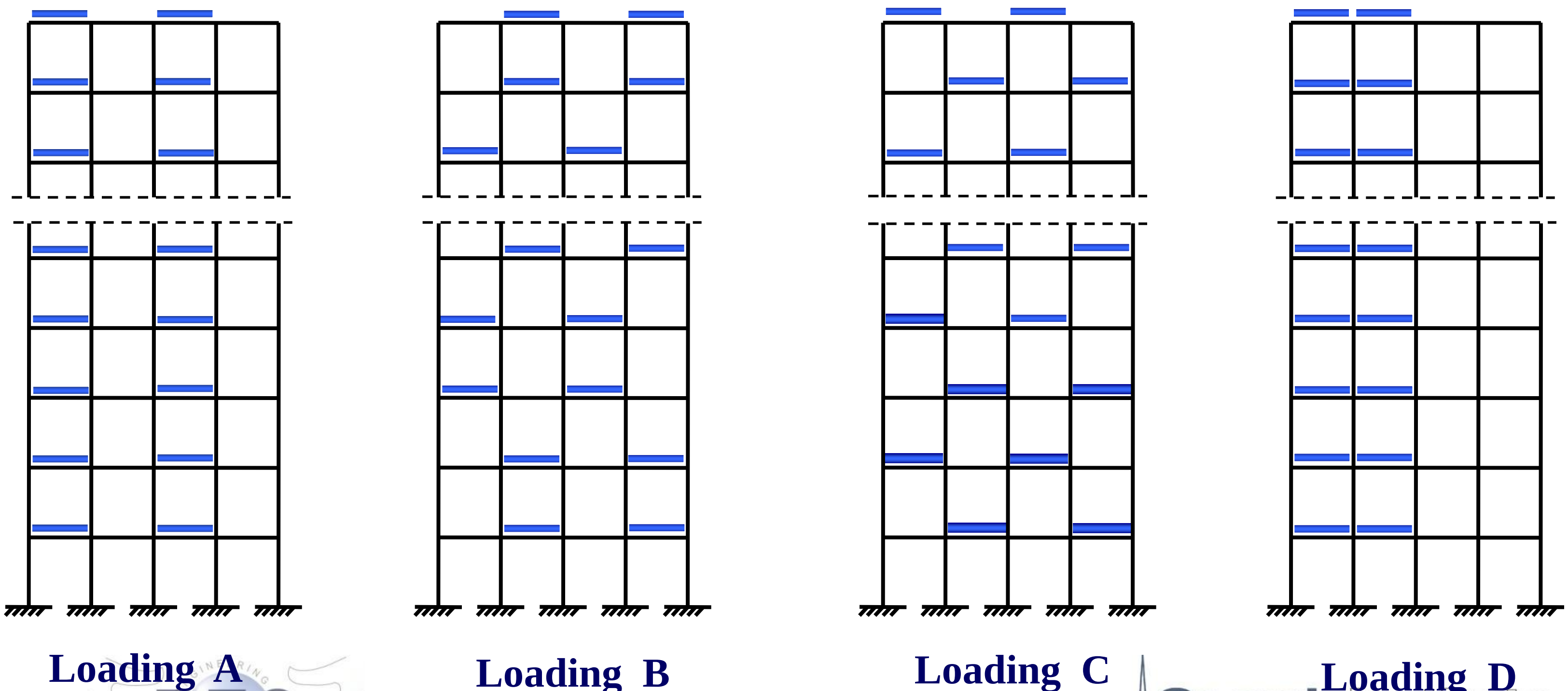




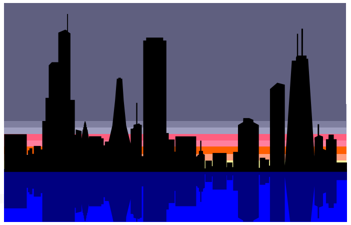
Examples – Load Uncertainty



- Four-bay forty-story frame



Examples – Load Uncertainty



- **Four-bay forty-story frame**

Total number of floor load patterns

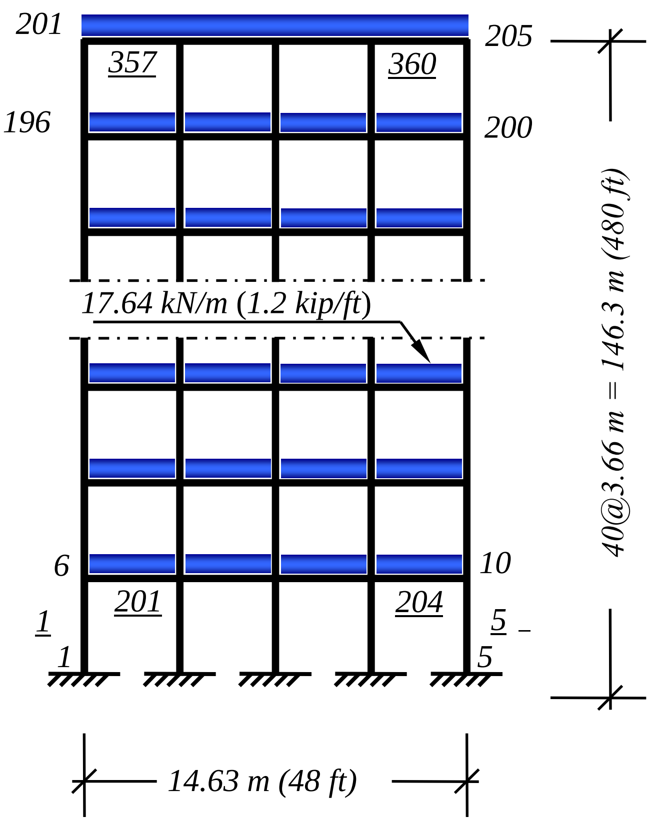
$$2^{160} = 1.46 \times 10^{48}$$

If one were able to calculate

10,000 *patterns / s*

there has not been sufficient time since the creation of the universe **(4-8)** billion years ? to solve all load patterns for this simple structure

Material *A36*, Beams *W24* x *55*, Columns *W14* x *398*



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Examples – Load Uncertainty

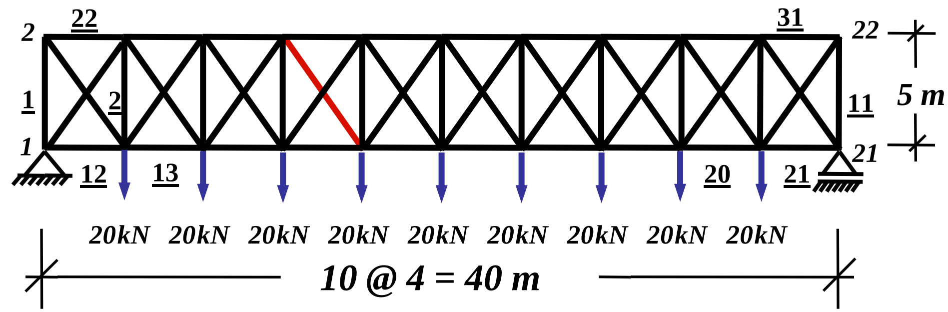
- Four-bay forty-story frame

Four bay forty floor frame - Interval solutions for shear force and bending moment of first floor columns

Elements		1		2		3	
Nodes		1	6	2	7	3	8
Combination solution		Total number of required combinations = $1.461501637 \times 10^{48}$					
Interval	Axial force (kN)	[-2034.5, 185.7]		[-2161.7, 0.0]		[-2226.7, 0.0]	
solution	Shear force (kN)	[-5.1, 0.9]		[-5.8, 5.0]		[-5.0, 5.0]	
	Moment (kN m)	[-10.3, 4.5]	[-15.3, 5.4]	[-10.6, 9.3]	[-17, 15.2]	[-8.9, 8.9]	[-16, 16]

Examples – Load Uncertainty

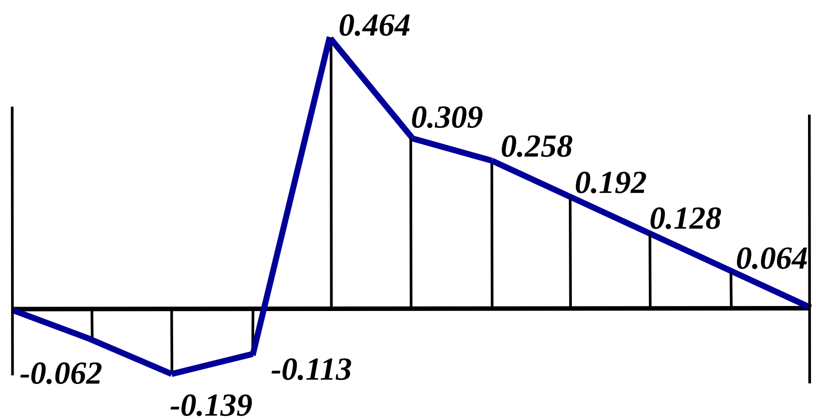
- **Ten-bay truss**



A = 0.006 m²

E = 2.0 ×10⁸ kPa

F = [-4.28, 28.3] kN

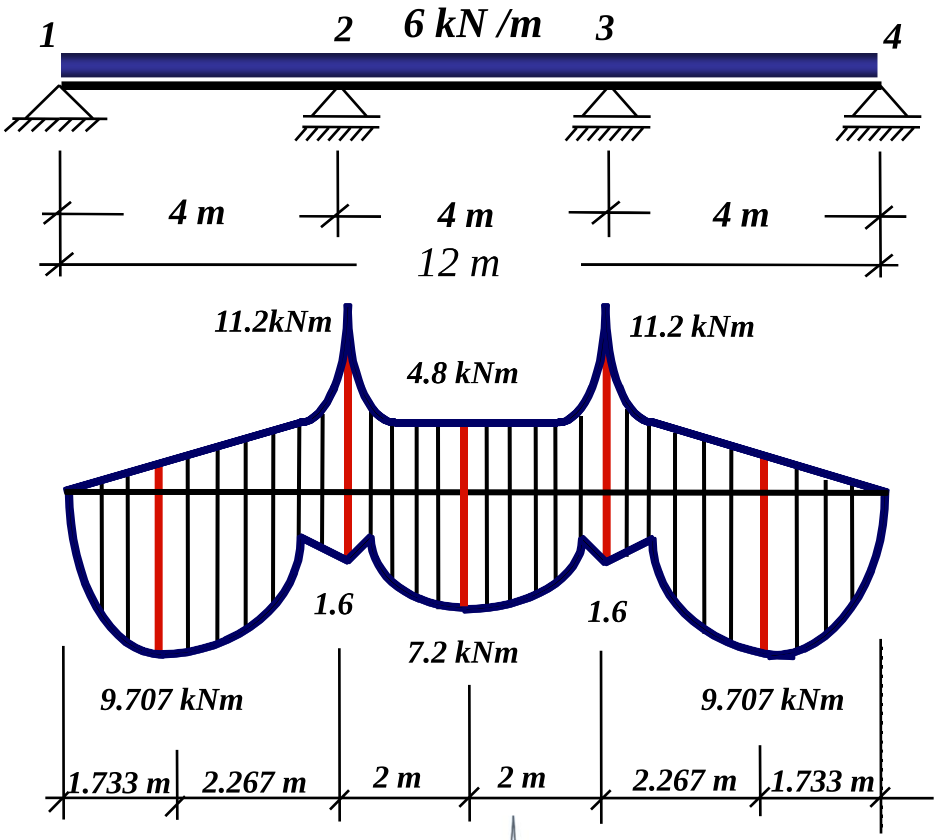


F_{min} = -(0.062+0.139+0.113) 20= **-4.28 kN**

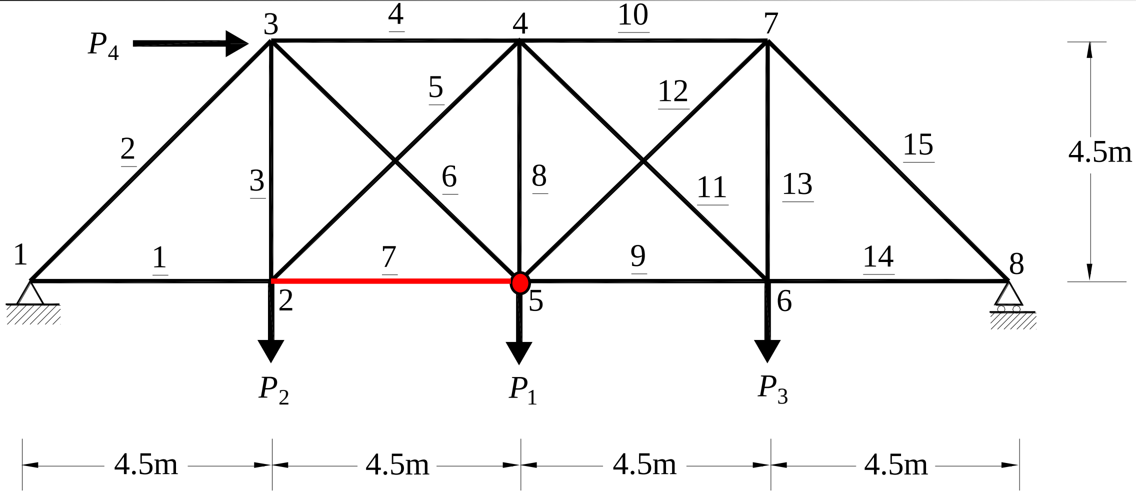
F_{max} = (0.464+0.309+0.258+0.192+0.128+0.064) 20 =**28.3 kN**

Examples – Load Uncertainty

- Three-Span Beam



Truss structure



A_1, A_2, A_3, A_4, A_5, A_6: [9.95, 10.05] cm^2 (1% uncertainty)

For all other members: [5.97, 6.03] cm^2 (1% uncertainty)

Modulus of elasticity for all members: 200,000 MPa

$\boldsymbol{p}_1 = [190, 210]$ kN, $\boldsymbol{p}_2 = [95, 105]$ kN

$\boldsymbol{p}_3 = [95, 105]$ kN, $\boldsymbol{p}_4 = [85.5, 94.5]$ kN (10% uncertainty)

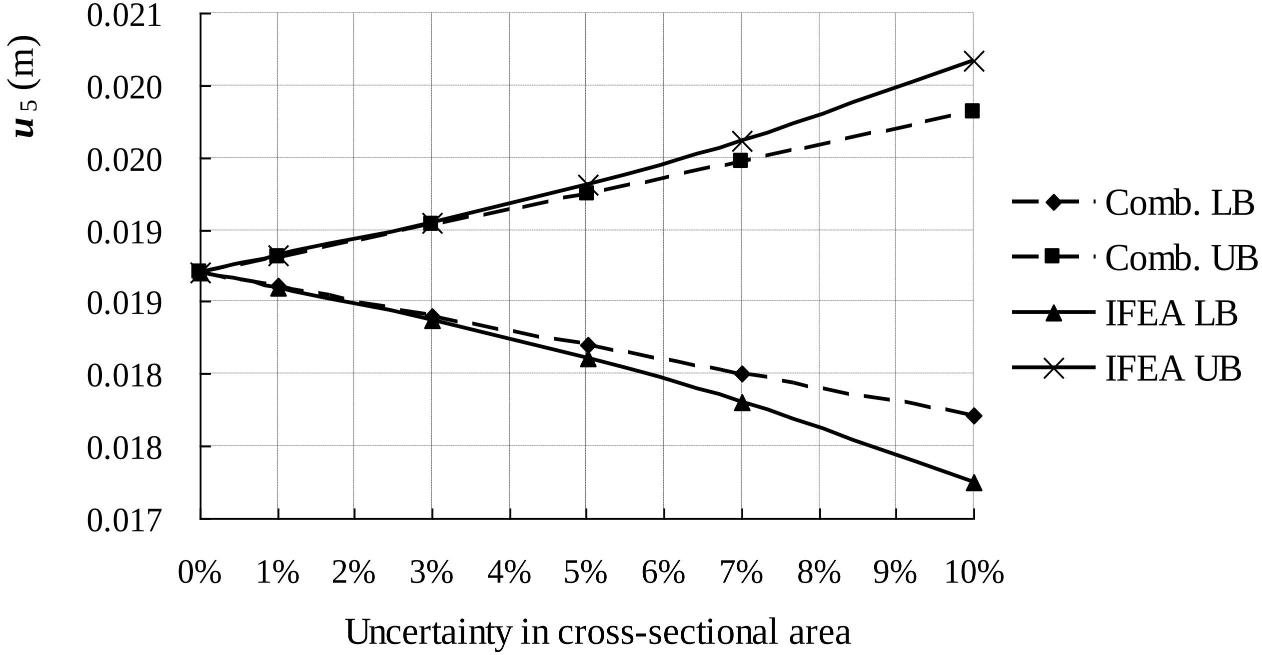
Truss structure - results

Table: results of selected responses

Method	u_5(LB)	u_5(UB)	N_7(LB)	N_7(UB)
Combinatorial	0.017676	0.019756	273.562	303.584
Naïve IFEA	– 0.011216	0.048636	– 717.152	1297.124
δ	163.45%	146.18%	362%	327%
Present IFEA	0.017642	0.019778	273.049	304.037
δ	0.19%	0.11%	0.19%	0.15%

unit: u_5 (m), N_7 (kN). LB: lower bound; UB: upper bound.

Truss structure – results



- for moderate uncertainty ($\leq$ 5%), very sharp bounds are obtained
- for relatively large uncertainty, reasonable bounds are obtained

  in the case of 10% uncertainty:

  Comb.: $\boldsymbol{u}_5$ = [0.017711,0.019811], IFEM: $\boldsymbol{u}_5$ = [0.017252,0.020168]

  (relative difference: 2.59%, 1.80% for LB, UB, respectively)

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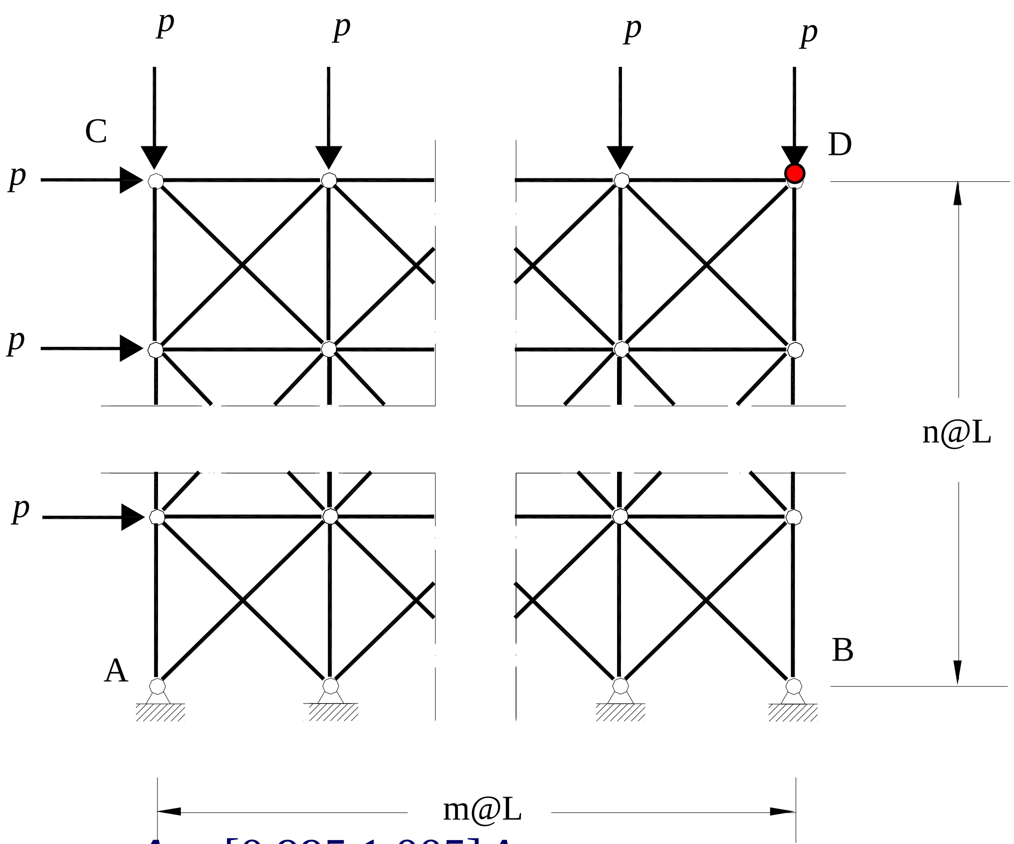
Truss with a large number of interval variables



$$\boldsymbol{A}_i = [0.995, 1.005] A_0,$$

$$\boldsymbol{E}_i = [0.995, 1.005] E_0 \quad \text{for } i = 1, \dots, N_e$$

story × bay	N_e	N_v
3×10	123	246
4×12	196	392
4×20	324	648
5×22	445	890
5×30	605	1210
6×30	726	1452
6×35	846	1692
6×40	966	1932
7×40	1127	2254
8×40	1288	2576

Scalability study

vertical displacement at right upper corner (node D): $\boldsymbol{v}_D = \boldsymbol{a}\dfrac{PL}{E_0 A_0}$

Table: displacement at node D

Story × bay	Sensitivity Analysis		Present IFEA				
	LB*	UB *	LB	UB	δ_{LB}	δ_{UB}	wid/d_0
3×10	2.5143	2.5756	2.5112	2.5782	0.12%	0.10%	2.64%
4×20	3.2592	3.3418	3.2532	3.3471	0.18%	0.16%	2.84%
5×30	4.0486	4.1532	4.0386	4.1624	0.25%	0.22%	3.02%
6×35	4.8482	4.9751	4.8326	4.9895	0.32%	0.29%	3.19%
7×40	5.6461	5.7954	5.6236	5.8166	0.40%	0.37%	3.37%
8×40	6.4570	6.6289	6.4259	6.6586	0.48%	0.45%	3.56%

$\delta_{LB} = |LB - LB^*| / LB^*$, $\delta_{LB} = |UB - UB^*| / UB^*$, $\delta_{LB} = (LB - LB^*) / LB^*$

Efficiency study

Table: CPU time for the analyses with the present method (unit: seconds)

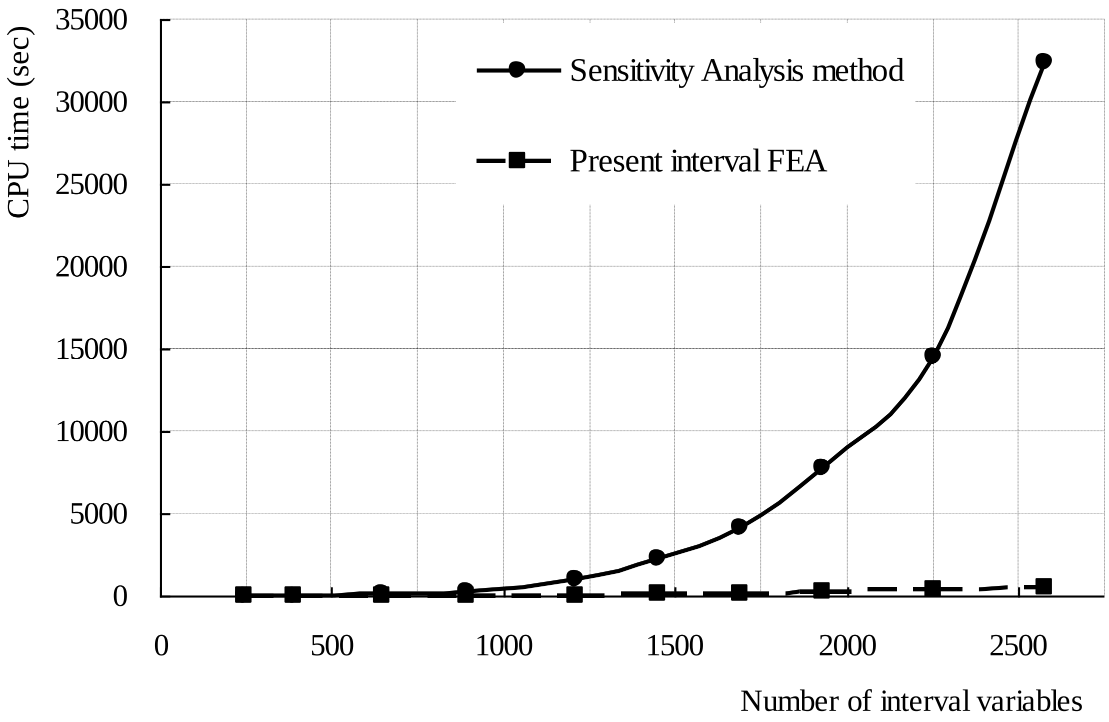
Story×bay	N_v	Iteration	t_i	t_r	t	t_i/t	t_r/t
3×10	246	4	0.14	0.56	0.72	19.5%	78.4%
4×20	648	5	1.27	8.80	10.17	12.4%	80.5%
5×30	1210	6	6.09	53.17	59.70	10.2%	89.1%
6×35	1692	6	15.11	140.23	156.27	9.7%	89.7%
7×40	2254	6	32.53	323.14	358.76	9.1%	90.1%
8×40	2576	7	48.454	475.72	528.45	9.2%	90.0%

t_i : iteration time, t_r : CPU time for matrix inversion, t : total comp. CPU time

- majority of time is spent on matrix inversion

Efficiency study

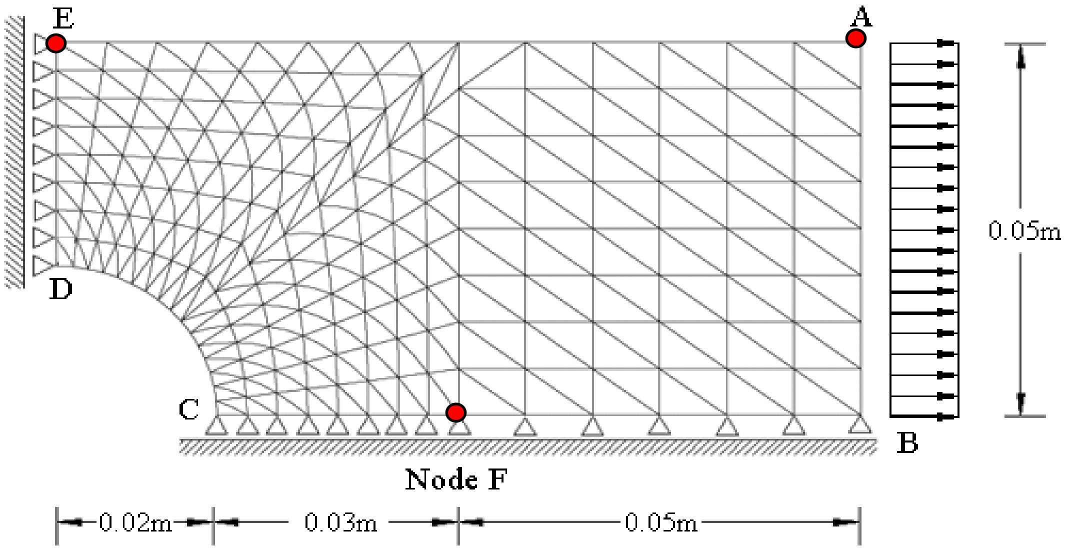
Computational time: a comparison of the senstitivity analysis method and the present method



Computational time (seconds)

N_v	Sens.	Present
246	1.06	0.72
648	64.05	10.17
1210	965.86	59.7
1692	4100	156.3
2254	14450	358.8
2576	32402	528.45
	9 hr	9 min

Plate with quarter-circle cutout



thickness: 0.005m

Possion ratio: 0.3

load: 100kN/m

modulus of elasticity:

$\boldsymbol{E}$ = [199, 201]GPa

number of element: 352

element type: six-node isoparametric quadratic triangle

results presented: $\boldsymbol{u}_A$, $\boldsymbol{v}_E$, $\boldsymbol{\sigma}_{xx}$ and $\boldsymbol{\sigma}_{yy}$ at node F

Plate with quarter-circle cutout

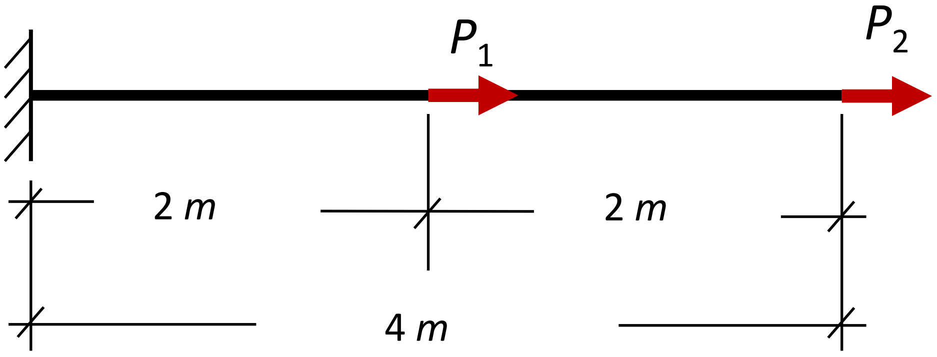
Case 1: the modulus of elasticity for each element varies independently in the interval [199, 201] GPa.

Table: results of selected responses

Response	Monte Carlo sampling*		Present IFEA	
	LB	UB	LB	UB
$\boldsymbol{u}_A$ (10^{-5} m)	1.19094	1.20081	1.18768	1.20387
$\boldsymbol{v}_E$ (10^{-5} m)	−0.42638	−0.42238	−0.42894	−0.41940
$\boldsymbol{\sigma}_{xx}$ (MPa)	13.164	13.223	12.699	13.690
$\boldsymbol{\sigma}_{yy}$ (MPa)	1.803	1.882	1.592	2.090

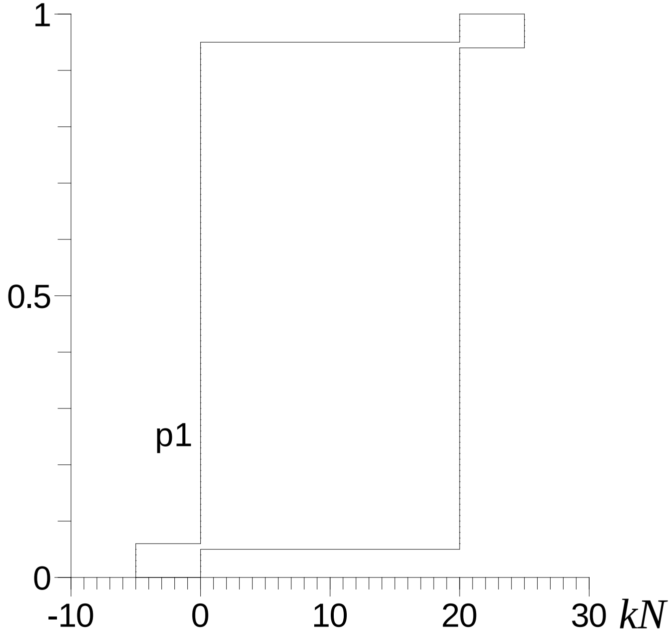
*10^6 samples are made.

Imprecise Probability

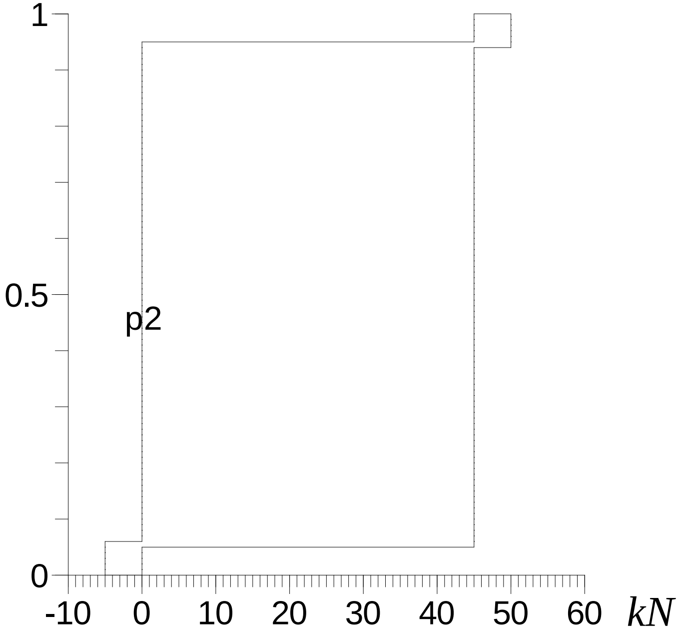


Two-bar truss

Imprecise Probability

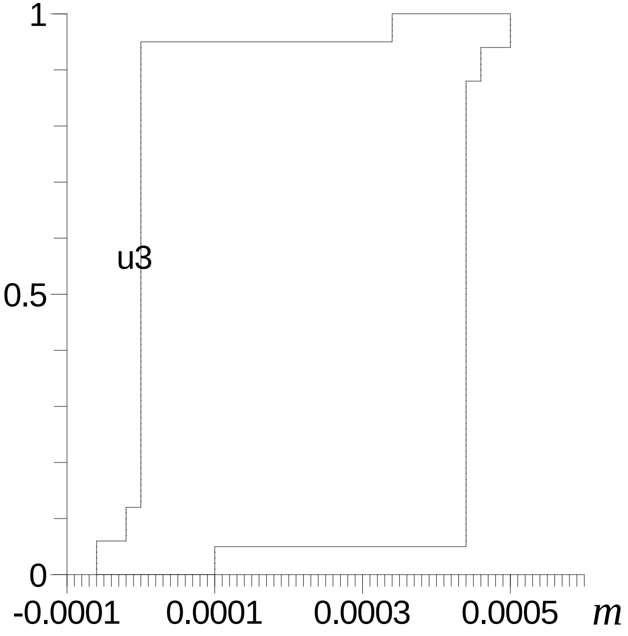


P-Box for P_1



P-Box for P_2

Imprecise Probability



P-Box for displacement of the bar end
obtained using FE analysis and Risk Calc (Ferson)

Outline

- Introduction
- Interval Arithmetic
- Interval Finite Elements
- Element-By-Element
- Examples
- Conclusions

Conclusions

- Development and implementation of IFEM
  - uncertain material, geometry and load parameters are described by interval variables
  - interval arithmetic is used to guarantee an enclosure of response
- Enhanced dependence problem control
  - use Element-By-Element technique
  - use the penalty method or Lagrange multiplier method to impose constraints
  - modify and enhance fixed point iteration to take into account the dependence problem
  - develop special algorithms to calculate stress and element nodal force

Conclusions

- The method is generally applicable to linear static FEM, regardless of element type
- Evaluation of the present method
  - Rigorousness: in all the examples, the results obtained by the present method enclose those from the alternative methods
  - Accuracy: sharp results are obtained for moderate parameter uncertainty (no more than 5%); reasonable results are obtained for relatively large parameter uncertainty (5%~10%)

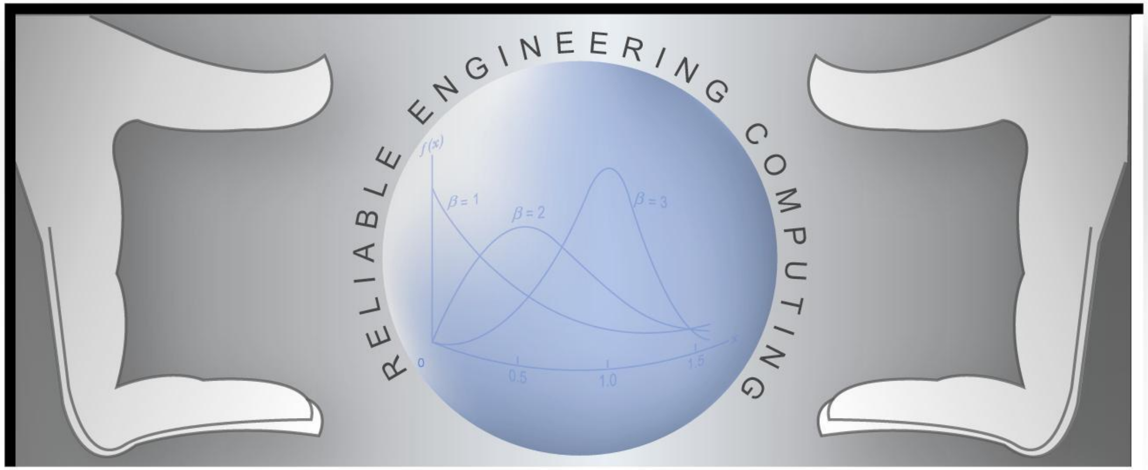
Conclusions

- Scalability: the accuracy of the method remains at the same level with increase of the problem scale
- Efficiency: the present method is significantly superior to the conventional methods such as the combinatorial, Monte Carlo sampling, and sensitivity analysis method

- IFEM forms a basis for generalized models of uncertainty in engineering
- The present IFEM represents an efficient method to handle uncertainty in engineering applications

Center for Reliable Engineering Computing (REC)

Thank You



We handle computations with care