Inner and Outer Approximation of Functionals

coming from static analysis

using

Generalized Affine Forms

Eric Goubault and Sylvie Putot

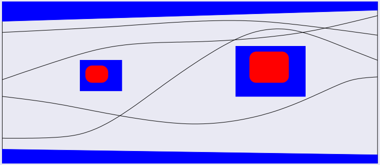
CEA-LIST, MEASI (ModElisation and Analysis of Systems in Interaction)

SCAN 2008, El Paso, TX

Context

Static analysis of programs

- Find outer-approximation of sets of reachable values of variables at some program points
- To ensure absence of runtime errors typically



Example

```
float x;
x=[0,1];          [1]
while (x<=1) {  [2]
  x = x-0.5*x;  [3]
}                 [4]
```

$$x_1 = [0,1]$$
$$x_2 =]-\infty,1] \cap (x_1 \cup x_3)$$
$$x_3 = x_2 - 0.5x_2$$
$$x_4 =]1,\infty[\cap x_2$$

(final smallest invariant: $x_2 \in [0,1]$, $x_4 = \emptyset$)

Motivation for this talk

Proof of good behaviour

- Need for tight and correct outer approximations
  - First part of the talk: How do we find invariant sets? How do we ensure correctness?
  - Based on affine forms - concentrate on real values first

But how pessimistic are the results?

- Joint use of inner- and outer-approximations to characterize the quality of analysis results
  - Inner-approximation: sets of values for the variables, that are sure to be reached for some inputs in the specified ranges.
  - (Second part of the talk) Use of affine forms with generalized intervals as coefficients

Affine Arithmetic for real numbers

Originally: Comba, de Figueiredo and Stolfi 1993

- A variable x is represented by an affine form $\hat{x}$:

$$\hat{x} = x_0 + x_1\varepsilon_1 + \ldots + x_n\varepsilon_n,$$

  where $x_i \in \mathbb{R}$ and ε_i are independent symbolic variables with unknown value in $[-1, 1]$.
  - $x_0 \in \mathbb{R}$ is the *central value* of the affine form
  - the coefficients $x_i \in \mathbb{R}$ are the *partial deviations*
  - the ε_i are the *noise symbols*
- The sharing of noise symbols between variables expresses *implicit dependency*

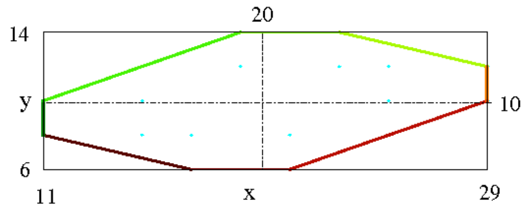
On top of that...

We want a notion of union (and intersections - outside the scope of this talk) of affine forms since we want to compute invariant forms of particular dynamical systems (programs).

They form sub-polyhedric relations

Concretization is a center-symmetric convex polytope

$$\hat{x} = 20 - 4\varepsilon_1 + 2\varepsilon_3 + 3\varepsilon_4$$
$$\hat{y} = 10 - 2\varepsilon_1 + \varepsilon_2 - \varepsilon_4$$



Define...

$$\gamma(\hat{x}) = [\alpha_0^x - \|\hat{x}\|_1, \alpha_0^x + \|\hat{x}\|_1]$$

where $\|\hat{x}\|_1 = \sum_{i=1}^{\infty} |\alpha_i^x|$, (finite, or ℓ_1-convergence)
Also define joint concretisation.

Affine Arithmetic for over-approximation (some functions)

Assignment

of a variable x whose value is given in a range $[a, b]$ at label i, introduces a noise symbol ε_i :

$$\hat{x} = \frac{(a+b)}{2} + \frac{(b-a)}{2}\varepsilon_i.$$

Addition

$$\hat{x} + \hat{y} = (\alpha_0^x + \alpha_0^y) + (\alpha_1^x + \alpha_1^y)\varepsilon_1 + \ldots + (\alpha_n^x + \alpha_n^y)\varepsilon_n$$

For example, with real (exact) coefficients , $f - f = 0$.

Multiplication

creates a new noise term (can do better):

$$\hat{x} \times \hat{y} = \alpha_0^x\alpha_0^y + \sum_{i=1}^{n}(\alpha_i^x\alpha_0^y + \alpha_i^y\alpha_0^x)\varepsilon_i + \left(\sum_{i=1}^{n}|\alpha_i^x|.|\sum_{i=1}^{n}|\alpha_i^y|\right)\varepsilon_{n+1}.$$

Interpretation of unions?

How do we compute...?

...as an affine form $\hat{z}$ the union of for instance:

$$\begin{array}{rcl} \hat{x} & = & 3 + \varepsilon_1 + 2\varepsilon_2 \\ \hat{y} & = & 1 - 2\varepsilon_1 + \varepsilon_2 \end{array}$$

Problem

- Easy geometric interpretation of union but difficult to find a good notion of "optimal" affine form representing a union
- Unions are some form of non-linear operations
- Our choice: distinguish a noise symbol ϵ_U for taking care of uncertainties due to unions (and intersections)

Join operation (see also Goubault/Putot 2008 [4])

Define $z = x \cup y$ by:

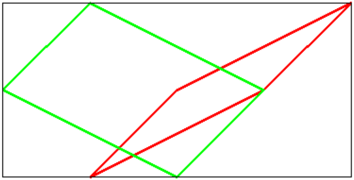
- $$\begin{cases} \alpha_0^z = mid(\gamma(\hat{x}) \cup \gamma(\hat{y})) \\ \alpha_i^z = \underset{\alpha_i^x \wedge \alpha_i^y \leq \alpha \leq \alpha_i^x \vee \alpha_i^y}{\text{argmin}} |\alpha|, \ \forall i \geq 1 \\ \beta^z = \sup \gamma(\hat{x}) \cup \gamma(\hat{y}) - \alpha_0^z - \|z\|_1 \end{cases}$$
- Intuitively, we keep in the union the minimal common dependencies, the "rest" being put as a coefficient to ϵ_U
- Meet similar...

Where...("minimal dependency")

$$\underset{u \wedge v \leq \alpha \leq u \vee v}{\text{argmin}} |\alpha| = \{\alpha \in [u \wedge v, u \vee v], |\alpha| \text{ minimal}\}$$

Example - again

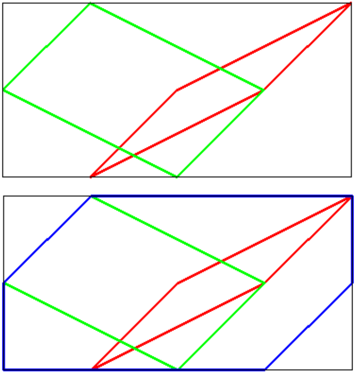
$$\hat{x} = 3 + \varepsilon_1 + 2\varepsilon_2$$
$$\hat{y} = 1 - 2\varepsilon_1 + \varepsilon_2$$
$$\hat{u} = \varepsilon_1 + \varepsilon_2$$



Example - again

$$\hat{x} = 3 + \varepsilon_1 + 2\varepsilon_2$$
$$\hat{y} = 1 - 2\varepsilon_1 + \varepsilon_2$$
$$\hat{u} = \varepsilon_1 + \varepsilon_2$$

$$\hat{x} \cup \hat{y} = 2 + \varepsilon_2 + 3\varepsilon_U$$



(Note that $\gamma(\hat{z}) = [-2, 6] = \gamma(\hat{x}) \cup \gamma(\hat{y})$)

Example of an invariant for a simple dynamical system/program

Consider:

$$
\begin{aligned}
x_i &= f(e_i, e_{i-1}, e_{i-2}, x_{i-1}, x_{i-2}) \\
&= 0.7e_i - 1.3e_{i-1} + 1.1e_{i-2} + 1.4x_{i-1} - 0.7x_{i-2}
\end{aligned}
$$

where e_i are independent inputs between 0 and 1.

Invariant set computation

We use Kleene iteration:

Compute

$$\hat{x}_i = \hat{x}_{i-1} \cup f(e_i, e_{i-1}, e_{i-2}, \hat{x}_{i-1}, \hat{x}_{i-2})$$

(in fact, we iterate f a little bit, by a factor k)

Invariant set

Results

- (k=5) we reach the over-approximation of the enclosure: [-1.6328,3.2995]
- (k=16) we reach [-1.3,2.8244] (in 18 iterations without widening)
- The smallest enclosure is actually [-1.121240...,2.824318...]

Note that this is not limited to independent inputs, or independent initial conditions.
For instance, if all the *inputs over time are equal* to an unknown number between 0 and 1, the final invariant found with k=16 has *concretization [-0.1008,2.3298]*.

Criteria for correctness

Replace concrete variables x_i and functions f by affine forms $\hat{x}_i$...?

[1] Range of individual variables

Given expressions $y_1 = e_1(x_1, \ldots, x_n), \ldots y_m = e_m(x_1, \ldots, x_n)$ depending on variables $x_1, \ldots, x_n$, ensure that $\gamma(\hat{y}_k)$ contains all concrete values y_k for all possible values of the x_j

[2] Joint range, given a fixed set of variables and expressions

Same but for the joint concretisation (as a zonotope) $\gamma(\hat{y}_1, \ldots, \hat{y}_m)$

[3] Future evaluations (or global consistency)

We want that for all expressions f, the range of $\hat{f}(\hat{y}_1, \ldots, \hat{y}_m)$ contains all concrete values $f(y_1, \ldots, y_m)$

Clearly... [3]⇒[2]⇒[1]

Converse?

Correctness?

Take (example by Kolev 2007)

$$
\begin{aligned}
\hat{x} &= 10 + 5\epsilon_1 + 3\epsilon_2 \\
\hat{y} &= 10 - 2\epsilon_1 + \epsilon_3 \\
\hat{z} &= 92 + 31\epsilon_1 + 21\epsilon_2 + 2\epsilon_3 + 16\epsilon_4 \quad \text{Kolev multiplication}
\end{aligned}
$$

Question:

Is $\hat{z}$ a good model for outer-approximating $\hat{x}\hat{y}$?

Correctness?

Take (example by Kolev 2007)

$$\hat{x} = 10 + 5\epsilon_1 + 3\epsilon_2$$
$$\hat{y} = 10 - 2\epsilon_1 + \epsilon_3$$
$$\hat{z} = 92 + 31\epsilon_1 + 21\epsilon_2 + 2\epsilon_3 + 16\epsilon_4 \quad \text{Kolev multiplication}$$

Question:

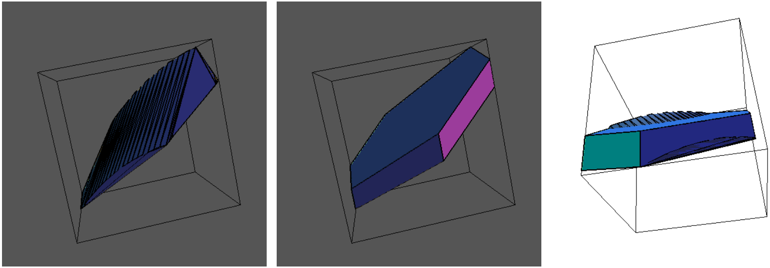
Is $\hat{z}$ a good model for outer-approximating $\hat{x}\hat{y}$?

Here

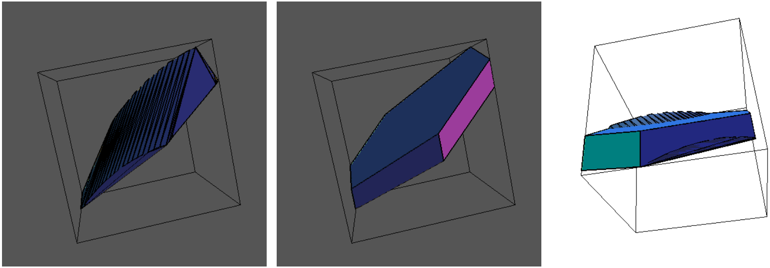
$$\gamma(\hat{z}) = [22, 162]$$

which is a correct range (and optimal) for the multiplication
We have criterion [1] (of course, this was designed for it!)

Joint range



Joint range



So we do not have [2]...

...Nor [3] (of course!)...

Consider (Khalil Ghorbal)

$$\begin{aligned} \hat{t} &= -4\hat{x} + 0.8\hat{z} - 79 \\ &= -45.4 + 4.8\epsilon_1 + 4.8\epsilon_2 + 1.6\epsilon_3 + 12.8\epsilon_4 \quad \in [-69.4, -21.4] \end{aligned}$$

But for $\epsilon_1 = 0$, $\epsilon_2 = 1$ and $\epsilon_3 = 1$,

$$x = 13,\ y = 11,\ z = 143$$

so $t = -16.6 > -21.4$!

...Nor [3] (of course!)...

Consider (Khalil Ghorbal)

$$\begin{aligned}\hat{t} &= -4\hat{x} + 0.8\hat{z} - 79 \\ &= -45.4 + 4.8\epsilon_1 + 4.8\epsilon_2 + 1.6\epsilon_3 + 12.8\epsilon_4 \quad \in [-69.4, -21.4]\end{aligned}$$

But for $\epsilon_1 = 0$, $\epsilon_2 = 1$ and $\epsilon_3 = 1$,

$$x = 13,\ y = 11,\ z = 143$$

so $t = -16.6 > -21.4$!

But...

...there are other multiplications for [3] (SDP based, to appear)

Also...[2]$\not\Rightarrow$[3]...

Consider...

$$\begin{cases} \hat{x} &= \epsilon_1 \\ \hat{y} &= \epsilon_2 \\ \hat{z} &= f(\hat{x},\hat{y}) = x + y - \epsilon_4 \\ &= \epsilon_1 + \epsilon_2 - \epsilon_4 \\ & \in [-3,3] \end{cases} \quad \begin{cases} \hat{x}' &= -\epsilon_1 \\ \hat{y}' &= \frac{1}{2}(\epsilon_3 + \epsilon_4) \\ \hat{z}' &= f(\hat{x}',\hat{y}') = x' + y' - \epsilon_4 \\ &= -\epsilon_1 + \frac{1}{2}(\epsilon_3 - \epsilon_4) \\ & \in [-2,2] \end{cases}$$

Clearly...

The joint concretisations of $(\hat{x},\hat{y})$ and of $(\hat{x}',\hat{y}')$ are the same (but with different dependencies), whereas the same future evaluation f does not give the same range on $(\hat{x},\hat{y})$ and on $(\hat{x}',\hat{y}')$

Partial conclusion

Correctness

- [3]⇒[2]⇒[1] but [1]$\not\Rightarrow$[2]$\not\Rightarrow$[3]
- [3] is definitely necessary when functionals to be evaluated are discovered along the way (as in static analysis)

Remark on union

- Partial order relation $\hat{x} \preceq \hat{y}$ if all future evaluations using $\hat{x}$ instead of $\hat{y}$ have smaller concretisation (can be characterized in a simpler manner see also Goubault/Putot 2008 [4])
- Our union operator is a minimal upper bound (under some conditions) for this order, reflecting some form of optimality under correctness criterion [3]

Partial conclusion

Correctness

- [3]$\Rightarrow$[2]$\Rightarrow$[1] but [1]$\not\Rightarrow$[2]$\not\Rightarrow$[3]
- [3] is definitely necessary when functionals to be evaluated are discovered along the way (as in static analysis)

Remark on union

- Partial order relation $\hat{x} \preceq \hat{y}$ if all future evaluations using $\hat{x}$ instead of $\hat{y}$ have smaller concretisation (can be characterized in a simpler manner see also Goubault/Putot 2008 [4])
- Our union operator is a minimal upper bound (under some conditions) for this order, reflecting some form of optimality under correctness criterion [3]

What about inner-approximations?

Inner-approximations (see also Goubault/Putot 2007 [3])

Principle

- Use more general dependency coefficients
  - $\check{x} = \sum_{i=1}^{n} [a_i, b_i]\varepsilon_i$ (+possibly generalized interval symbols)
  - *Generalized intervals* : $\mathbf{x} = [\underline{x}, \overline{x}]$, possibly with $\underline{x} \geq \overline{x}$.

First, recap of modal intervals

- dual $\mathbf{x} = \mathbf{x}^* = [\overline{x}, \underline{x}]$ and pro $\mathbf{x} = [min(\underline{x}, \overline{x}), max(\underline{x}, \overline{x})]$.
- $\mathbf{x}$ is *proper* (in $\mathbb{IR}$) if $\underline{x} \leq \overline{x}$, otherwise *improper*
- Kaucher arithmetic extending classical interval arithmetic
  - For instance same addition
  - But $[1, 2] * [1, -1] = [1, -1]$ whereas $[1, 2] *$ pro $[1, -1] = [2, -2]$

Modal intervals/Quantifiers (à la Goldsztejn 2005 [1])

Classical over-approximated interval computation

All intervals are proper
$(\forall x \in \mathbf{x})\,(\exists z \in \mathbf{z})\,(f(x) = z)$.

- Let $f(x) = x^2 - x$, then $f([2,3]) = [2,3]^2 - [2,3] = [1,7]$ is interpreted as $(\forall x \in [2,3])\,(\exists z \in [1,7])\,(f(x) = z)$.

Inner-approximated computation

All intervals are improper
$(\forall z \in \text{pro}\ \mathbf{z})\,(\exists x \in \text{pro}\ \mathbf{x})\,(f(x) = z)$.

- Application scope is limited to expressions with no dependency between sub-expressions
- An inner-approximation of $f(x) = x^2 - x$ for $x \in [2,3]$ cannot be thus computed

Inner- and outer-approximations

Example: inner multiplication (using Goldsztejn 2005 [1])

Let $\hat{x}$ and $\hat{y}$ be two affine forms (real coeff.) and $\mathbf{z} = x \times y$

- An inner-approximation is

$$\check{z} = \alpha_0^x \alpha_0^y + \sum_{i=1}^{n} (\alpha_i^x \alpha_0^y + \alpha_i^y \alpha_0^x) \varepsilon_i + \left(\sum_{j=1}^{n} (\alpha_i^x \alpha_j^y + \alpha_i^y \alpha_j^x) \boldsymbol{\varepsilon}_\mathbf{j} \right) \varepsilon_i$$

  - over-approximation of dependencies,
  - $\boldsymbol{\alpha}_\mathbf{i}^\mathbf{z}$ contains the tangent $\frac{\partial z}{\partial \varepsilon_i}$

An outer-approximation is

$$\hat{z} = \alpha_0^x \alpha_0^y + \sum_{i=1}^{n} (\alpha_i^x \alpha_0^y + \alpha_i^y \alpha_0^x) \varepsilon_i + \left(\sum_{i=1}^{n} |\alpha_i^x| . | \sum_{i=1}^{n} |\alpha_i^y| \right) \varepsilon_{n+1},$$

with a new noise symbol ε_{n+1} : over-approximation by loss of dependency between linear terms and the non linear term.
The purely affine part of the product is the same

Back to the example

Consider

$$f(x) = x^2 - x \text{ when } x \in [2,3] \text{ (real result } [2,6])$$

We find:

- $$\tilde{f}^{\varepsilon}(\varepsilon_1) = 3.75 + [1.5, 2.5]\varepsilon_1$$

  Inner-approximating concretization
  $3.75 + [1.5, 2.5][1, -1] = 3.75 + [1.5, -1.5] = [5.25, 2.25]$
  Outer-approximating concretization
  $3.75 + [1.5, 2.5][-1, 1] = 3.75 + [-2.5, 2.5] = [1.25, 6.25]$
- Affine arithmetic (over-approximation)

$$x^2 - x = [3.75, 4] + 2\varepsilon_1 \text{ (concretization } [1.75, 6])$$

Join and meet operations

Join

$$\check{z} = \check{x} \cup \check{y} = (\boldsymbol{\alpha_0^x} \cup \boldsymbol{\alpha_0^y}) + (\boldsymbol{\alpha_1^x} \cup \boldsymbol{\alpha_1^y})\varepsilon_1 + \ldots + (\boldsymbol{\alpha_n^x} \cup \boldsymbol{\alpha_n^y})\varepsilon_n.$$

Meet

If for $i \geq 0$, $\boldsymbol{\alpha_i^x} \cap \boldsymbol{\alpha_i^y} \neq \emptyset$, we can define an inner-approximation of the intersection by

$$\check{z} = \check{x} \cap \check{y} = (\boldsymbol{\alpha_0^x} \cap \boldsymbol{\alpha_0^y}) + (\boldsymbol{\alpha_1^x} \cap \boldsymbol{\alpha_1^y})\varepsilon_1 + \ldots + (\boldsymbol{\alpha_n^x} \cap \boldsymbol{\alpha_n^y})\varepsilon_n.$$

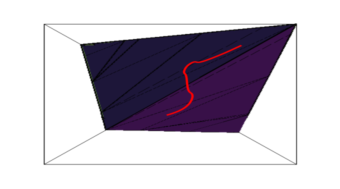
Otherwise, the result is $\perp$ (possible refinement by propagating instead the constraints induced on the ε_i).

Single inner-approximation versus joint inner-approximation versus future evaluations

Our joint concretization

The joint concretization has an a priori weak meaning

$$
\begin{aligned}
x_1 &= 5 + \varepsilon_1 \\
x_2 &= 2 + \varepsilon_2 \\
x_3 &= x_1 x_2 \\
&= 10 + [1,3]\varepsilon_1 + [4,6]\varepsilon_2 \\
& \underline{[5,15]} \subseteq [4,18] \subseteq \overline{[3,19]}
\end{aligned}
$$



$\forall z \in [5,15], \exists \epsilon_1, \epsilon_2,$
$z = x_1 x_2$

.

But we can prove...

...that our formulas agree with [1] but also make all future evaluations correct (criterion [3])

Joint inner range?

Using Goldsztejn/Jaulin 2008 [2] for joint concretization

Technical conditions ensure that both 2-dim boxes are included in the concrete joint range:

$$\begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 + \epsilon_1^* + 0\epsilon_2 \\ 10 + [1,3]\epsilon_1 + [4,6]\epsilon_2^* \end{pmatrix} = \begin{pmatrix} [4,6] \\ [7,13] \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 + \epsilon_1^* + 0\epsilon_2 \\ 2 + 0\epsilon_1 + \epsilon_2^* \end{pmatrix} = \begin{pmatrix} [4,6] \\ [1,3] \end{pmatrix}$$

So some surfaces are there inside the joint concretisation... but not possible to characterize a full 3D box inside...

Final conclusion

On correctness...

- For inner-approximations in our framework, criterion [2] is intractable in general:
  - for outer-approximations, still correct when losing dependencies
  - for inner-approximations, we have to outer-approximate dependencies
- The more rigid criterion [3] still applies!

We have a proven general inner-/outer- approximation calculus

- Of course, many details omitted ("splitting" for instance)

Perspectives

Can it be generalized to Taylor models?

Generalized *perturbed* affine forms

using $\epsilon_\cap$ symbols?

Floating-point and rounding error estimations

- Existing extension of the abstract domain (NSAD'05, SAS'06) for outer-approximation
- Problematic for inner-approximation

Faster-than-Kleene fixpoint computation

using policy iteration (CAV'05, ESOP'07)

Some references

[1] Alexandre Goldsztejn

Modal Intervals Revisited Part II: A Generalized Interval Mean-Value Extension HAL report number hal-00294222

[2] Alexandre Goldsztejn, Luc Jaulin

Inner Approximation of the Range of Vector-Valued Functions Reliable Computing (Springer), 2008

[3] Eric Goubault, Sylvie Putot

Under-Approximations of Computations in Real Numbers Based on Generalized Affine Arithmetic. SAS 2007

[4] Eric Goubault and Sylvie Putot

Perturbated affine arithmetic for invariant computation in numerical program analysis, arXiv:0807.2961, july 2008