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The algebra of many-valued quantities.

Von

Rosalind Cecily Young in Cambridge (England).

The following theory is one that I have recently adopted for the better treatment of theories involving limits[1]), where it has grown increasingly inconvenient to have to consider separately the upper or lower or other individual values of a numerical limit which is not unique[2]), for want

[1]) A preliminary treatment was embodied in my Dissertation for the Ph. D., Cambridge, "Foundations for the generalisation of the Theory of Stieltjes Integration etc. An n-dimensional treatment" (1929) and indicates the main features of the theory. Refinements introduced into the present exposition may be summarised as follows.

1. In the concept of *many-valuedness*, a symbol a (now a *quantity*), instead of being identified with a *set* (of values), is now conceived as having *any one* of a given set of values *collectively considered*,—in contradistinction to the idea of a variable, which assumes *individually considered values* in a given range which is generally fixed.—The exact nature of the concept, as of any mathematical entity, is best understood from the uses to which the concept is put, and in this case these are quite different from manipulations of sets.

2. By the introduction of the quantity having *no* values, $\mathring{o}$, the new *nought* (without prejudice to the "zero" (0) of our ordinary numberscheme), several simplifications are rendered possible; and *inter alia*

3. the definition of a *link* of two quantities (having the values common to both) as precisely complementary to that of their *union* (which has all the values of either); and the purer conception of the process of *levelling* (suppressing all values numerically $> K$).

4. The explicit definition of an *infinitesimal* also simplifies the exposition.

5. I absolutely exclude any reference to "infinite values", pending the precise definition and theory of such values, which will form the subject of a later paper. In accordance with this, the treatment of limits is that of *finite limits* throughout, i. e. concerns exclusively the finite values of limits, which may *or may not* constitute the *complete limits*. On this point, the present treatment is a good deal more precise than the original one.

[2]) The general idea of considering *all the limits*, and not merely upper and lower limits, seems to have been first utilized by W. H. Young in 1908: "Sulle due funzioni a più valori costituite dai limiti d'una funzione di variabile reale a destra ed a sinistra di ciascun punto", Rend. Acc. Lincei (5) 17, 582—87.

of accurate rules for their collective manipulation[3]). Applications of the theory will be published elsewhere; the theory, however, seems to be of sufficient interest in itself. As an illustration of the efficiency of the new instrument, the rules given by Theorem V (p. 283), and more generally by Theorems VI and VII, should be compared, also for elegance and precision, with the current inequalities (which they of course include):

$$\underline{\lim}_{m\to\infty} a_m + \underline{\lim}_{m\to\infty} b_m \leqq \underline{\lim}_{m\to\infty} (a_m + b_m) \leqq \underline{\lim}_{m\to\infty} a_m + \overline{\lim}_{m\to\infty} b_m \leqq \overline{\lim}_{m\to\infty} (a_m + b_m)$$

$$\leqq \overline{\lim}_{m\to\infty} a_m + \overline{\lim}_{m\to\infty} b_m,$$

and the corresponding ones for products.

1. Many-valuedness.

When a symbol $a, b, x, f(x)$, etc. represents *any one* of a given *set of numbers*, we say that it represents, or *is*, a (*finite*) *quantity*, and the given numbers are called its *values*.

In the particular case when there is only one number given, the quantity is said to be *one-valued* and is identified with the given number.

The necessary pendent of the notion of a quantity with more than one value is that of a quantity *without any values*; this then has to be classed with many-valued quantities in the same way as the null-set has to be classed with sets generally. We shall call it *nought*[4]) and denote it by

$$\mathring{o}.$$

A quantity with at least one value is therefore said to be *non-nought*.

A quantity with a bounded set of values only is said to be *strictly finite*. A quantity with positive values only is said to be *positive*, and if its values have a positive lower bound it is more specifically described as *strictly positive*. Similarly, a quantity with negative values only is said to be *negative*, and if its values have a negative upper bound, it is said to be *strictly negative*.

A quantity none of whose values is 0, is said to be *definite*. If it does not have values as near as we please to 0, it is said to be *strictly* definite.

If any of these properties belongs to a given one-valued quantity, it belongs to it of course strictly.

[3]) Such relations as appear in the paper of W. H. Young just quoted, e. g.

$$F_L(P) < n, \quad H_R(P) \leqq n, \quad G_R(P) + k < F_L(P) < H_R(P) - k,$$

for his many-valued (right- and left-hand limiting) functions, tacitly assume some rudiments of an algebra of many-valued quantities, though the relations in this case being of so simple a nature, there was no inconvenience or ambiguity in introducing them.

[4]) The ordinary 0 is called *zero*.

2. Relations.

Two quantities a and b are said to be *equal*, and we write

$$a = b,$$

if the set of values of a and that of b are identical; i. e. if every value of a is a value of b and vice versa,

A quantity a is said to be *included in* another b, and b is said to *include* a, and we write

$$a \subset b \quad \text{or} \quad b \supset a,$$

if every value of a is a value of b (but not necessarily the converse). In particular, for every quantity a,

$$\bar{o} \subset a.$$

For one-valued quantities, inclusion reduces of course to equality.

A quantity a is said to be *less than* another b, and b is said to be *greater than* a, and we write

$$a < b \quad \text{or} \quad b > a,$$

if each value of a is less than one of b *and* each value of b is greater than one of a. (In particular neither a nor b may be $\bar{o}$.)

The relation is said to be *strict* if it also holds between the upper bounds and between the lower bounds of the two sets of values; i. e. provided neither these upper bounds nor these lower bounds are equal.

A quantity a is said to be *superior to* b, and b is said to be *inferior to* a, and (for reasons which will at once appear) we write

$$a \nleqq b \quad \text{or} \quad b \ngeqq a,$$

whenever *either* some value of a is greater than *each* value of b, *or* some value of b is less than each value of a. (In particular, neither a nor b may be $\bar{o}$.)

To express the fact that a is *not superior to* b, and b *not inferior to* a, we write therefore

$$a \leqq b \quad \text{or} \quad b \geqq a.$$

This means then that either a or b is $\bar{o}$, or each value of a is $\leqq$ some value of b, *and* each value of b is $\geqq$ some value of a.

Similarly, a is *not less than* b, and b *not greater than* a, and we therefore write

$$a \nless b \quad \text{or} \quad b \ngtr a,$$

if either a or b is $\bar{o}$, or some value of a is $\geqq$ all values of b, *or* some value of b is $\leqq$ all values of a.

For one-valued quantities, "less than" and "inferior to" are equivalent phrases.

If a is less than b, it is *a fortiori* not superior to b; which is equivalent to saying that if a is superior to b, it is *a fortiori* not less than b; symbolically:

$$a < b \text{ implies } a \leqq b; \qquad a \nleqq b \text{ implies } a \nless b.$$

Two *strictly finite* quantities cannot be both less than and greater than one another; i. e.

$$a < b \text{ implies } a \ngtr b.$$

But they may very well be both superior and inferior to one another; the necessary and sufficient condition for

$$a \nleqq b \quad \text{and} \quad a \ngeqq b$$

to hold simultaneously is indeed merely that either a has all its values less than one, and greater than another, value of b, or b has all its values less than one, and greater than another, value of a. In particular, we cannot then have $a = b$; i. e. a many-valued quantity a cannot be superior or inferior to itself, so that we have always

$$a \leqq a \quad \text{and} \quad a \geqq a.$$

A given quantity a *may* moreover be neither superior nor inferior to another given quantity b, i. e.

$$a \leqq b \quad \text{and} \quad a \geqq b$$

may hold simultaneously without a and b being equal; a necessary and sufficient condition for this (if we omit the trivial case $a = \bar{o}$ or $b = \bar{o}$) is in fact that the set of values of a and the set of values of b should (without coinciding) have the same upper bound, included in both or in neither of the two sets, and the same lower bound, also included in both or in neither of the two sets.

If a is included in b, it cannot be *strictly* greater or less than b.

But it may still be either not superior or not inferior to b, or both; indeed a necessary and sufficient condition for

$$a \subset b \quad \text{and} \quad a \leqq b$$

to hold simultaneously is merely that the values of b are those of a together possibly with others not less than these; and that for

$$a \subset b \quad \text{and} \quad a \geqq b$$

to hold simultaneously is that the values of b are those of a, together possibly with others not greater than these.

Again, if a is included in b, it may also be either superior or inferior to b, or both; in fact, for

$$a \subset b \quad \text{and} \quad a \nleqq b$$

to hold simultaneously, it is necessary and sufficient that, besides all the values of a, the quantity b should have at least one value less than all these; and for

$$a < b \quad \text{and} \quad a \ngeq b$$

to hold simultaneously, it is necessary and sufficient that, besides all the values of a, the quantity b should have at least one value greater than all these.

3. Associated quantities.

Associated with any non-nought quantity a, we define, as far as existent (finite)

a) the *upper value*

$$\bar{a}$$

or upper bound of all the values of a;

the *lower value*

$$\underline{a}$$

or lower bound of all the values of a;

the *breadth*

$$\beta_a = \bar{a} - \underline{a}$$

or span of the set of values of a;

b) the $+$ *part*

$$a_+$$

or quantity equal to a when this is positive, and otherwise having as its values all the positive values of a and the value 0; (when a is one-valued a_+ is thus the larger of a and 0);

the $-$ *part*

$$a_-$$

equal to a when this is negative, and otherwise having all the negative values of a and the value 0; (if a is one-valued, a_- is thus the lesser of a and 0);

c) the *absolute magnitude*

$$|a|$$

or quantity having as its values the moduli of the values of a;

d) the *opposite*

$$-a$$

whose values are numerically the same as those of a, but with the opposite signs;

e) the *inverse*

$$\frac{1}{a}$$

(defined only when a is *definite*) whose values are the inverse $\frac{1}{\alpha}$ of the values α of a;

f) *limiting values*, or numerical limits of sequences of values of the quantity a;

the (*finite*) *frame*

$$\boxed{a}$$

or quantity having as its values all the values and all the limiting values of a; thus a one-valued quantity, and generally any quantity with only a finite number of values, is its own frame.

A quantity which is its own frame is said to be *closed* (*finitely*). If it is also strictly finite, it is said more specifically to be *completely closed*. This means that it has a closed set of values.

A quantity which is its own opposite is said to be *symmetrical*. The only one-valued symmetrical quantity is 0.

A quantity which is its own inverse is said to be *reciprocal*. The only one-valued reciprocal quantity is 1.

The opposite of $-a$ is clearly a; so is the inverse of $\frac{1}{a}$.

Each of the other associated quantities of a is the associated quantity of same name of itself. E. g.

$$\overline{(\bar{a})} = \bar{a}; \quad (a_+)_+ = a_+; \quad |(|a|)| = |a|.$$

In particular, the frame of a is always closed (finitely).

As regards the associated quantities of different names of each associated quantity of a, we have the following relations, whose proof is immediate.

Upper and lower values.

$$(\bar{a})_+ = \overline{(a_+)}, \quad (\bar{a})_- = \overline{(a_-)}, \quad (\underline{a})_+ = \underline{(a_+)}, \quad (\underline{a})_- = \underline{(a_-)};$$

$$\overline{(-a)} = -(\underline{a}), \quad \underline{(-a)} = -(\bar{a}); \quad \overline{\boxed{a}} = \bar{a}, \quad \underline{\boxed{a}} = \underline{a};$$

$$\overline{|a|} = \text{larger of} \begin{Bmatrix} |\bar{a}| \\ \text{and} \\ |\underline{a}| \end{Bmatrix},$$

and hence also

$$\beta_{|a|} \leqq \beta_a.$$

$+$ and $-$ parts.

$$(-a)_+ = -(a_-), \quad (-a)_- = -(a_+); \quad \boxed{a}_+ = \boxed{a_+}, \quad \boxed{a}_- = \boxed{a_-}.$$

Absolute magnitude.

$$|-a| = |a|, \quad |\boxed{a}| = \boxed{|a|}.$$

Frame.

$$\boxed{-a} = -\boxed{a}.$$

Interpreting the relations between two quantities in terms of their associated quantities, we have

A.

$$a \prec b$$

is equivalent to

$$-a \prec -b,$$

and to

$$\frac{1}{a} \prec \frac{1}{b}$$

when both these inverses are defined.

It implies moreover

$$|a| \prec |b|, \quad a_+ \prec b_+, \quad a_- \prec b_-, \quad \boxed{a} \prec \boxed{b},$$

and

$$\bar{a} \leqq \bar{b}, \quad \underline{a} \geqq \underline{b}, \quad \beta_a \leqq \beta_b.$$

B.

$$a < b$$

is equivalent to

$$-a > -b,$$

and to

$$\frac{1}{a} > \frac{1}{b}$$

when a, b are positive.

Moreover, if it holds *strictly*, it is equivalent to

$$\bar{a} < \bar{b}, \quad \underline{a} < \underline{b},$$

hence to

$$\boxed{a} < \boxed{b}.$$

C.

$$a \leqq b$$

is equivalent to

$$-a \geqq -b,$$

and to

$$\frac{1}{a} \geqq \frac{1}{b}$$

when a, b are positive.

Also it implies

$$\bar{a} \leqq \bar{b}, \quad \underline{a} \leqq \underline{b}$$

and is equivalent to this pair of relations if b includes its upper and a its lower value.

It implies further

$$a_+ \leqq b_+, \quad a_- \leqq b_-,$$

and

$$\boxed{a} \leqq \boxed{b}.$$

D. By taking complementaries of the above propositions, we obtain those relative to the other two types of relation.

We shall define all the associated quantities of *nought* as again *nought*. I. e.

$$\bar{o} = \underline{o} = \beta_o = o; \quad -o = o; \quad \frac{1}{o} = o;$$
$$|o| = o; \quad \boxed{o} = o;$$
$$o_+ = o; \quad o_- = o.$$

4. Operations.

The *sum*

$$a + b$$

of two non-nought quantities a and b is defined as the non-nought quantity having all, and only, the values which are sums of a value of a and a value of b. The *sum*

$$a + b + c$$

of three non-nought quantities, a, b and c is defined as the non-nought quantity having all, and only, the values which are sums of a value of a, one of b and one of c. And so for any number of quantities.

Similarly the *difference*

$$a - b$$

is defined as having all, and only, the values which are obtained by subtracting a value of b from a value of a, and is clearly the same as the sum of a and $-b$.

The *product*

$$a \cdot b$$

of two non-nought quantities a and b is similarly defined as having all, and only, the values which are products of a value of a by one of b. And correspondingly the product of any number of non-nought quantities is defined.

With the corresponding definition, the *ratio*

$$\frac{a}{b}$$

of a by b, defined only when b is definite, is seen to be the product of a and $\frac{1}{b}$.

The *associative* and *commutative laws* for addition, subtraction and multiplication in ordinary algebra obviously continue to hold, without any formal change; the *monotony laws*

$$\text{“}a < b \quad \text{implies} \quad (a + c) < (b + c)\text{”}$$

and

$$\text{“}a < b \quad \text{implies} \quad ac < bc \quad \text{when } c \text{ is positive”},$$

and the corresponding ones with $\leqq$ instead of $<$, are similarly unaltered. The new relation of inclusion furnishes us with a new type of law

$$\text{“}a \subset b \quad \text{implies} \quad (a+c) \subset (b+c)\text{”}$$

and

$$\text{“}a \subset b \quad \text{implies} \quad ac \subset bc\text{”},$$

called the *inclusion laws* for addition and multiplication.

Furthermore, we note that

if c is strictly finite,

$$c + d \subset c \quad \textit{implies} \quad d = 0;$$

if c is strictly finite and strictly positive or negative,

$$cd \subset c \quad \textit{implies} \quad d = 1.$$

For if

$$c + d \subset c,$$

and γ be any value of c, δ any value of d, $\gamma + \delta$ is also a value of c, and so are $\gamma + 2\delta$, $\gamma + 3\delta$, and generally $\gamma + N\delta$, for every integer N: so that if c is strictly finite, δ is necessarily 0.

Similarly if

$$cd \subset c,$$

and γ be any value of c, δ any value of d, then $\gamma\delta$, $\gamma\delta^2$ and generally $\gamma\delta^N$, are also values of c, for every integer N; so that if c is strictly finite, $|\delta|$ is necessarily $\leqq 1$, and if c is strictly positive or strictly negative, $|\delta|$ is necessarily $\geqq 1$. Hence q. e. d.

It is essential also to note that although

$$a - a \supset 0$$

always, $(a - a)$ is *only* $= 0$ *if a is one-valued.* From this and the above, we conclude at once that

$$a + b \subset c \quad \textit{implies} \quad a \subset c - b,$$

but for strictly finite c, the converse requires b to be one-valued.

The remark that $a - a = 0$ requires a to be one-valued is a particular case of the following:

A sum of given quantities is one-valued (if and) only if each of the given quantities is one-valued.

For if one of the quantities is not one-valued, and we choose any fixed value of each of the others, every value of the first quantity necessarily gives rise by addition with these to a different value of the sum.

Similarly,

A product of given quantities is one-valued (if and) only if each of the given quantities is one-valued, unless indeed one of these quantities is the number 0, when the product is also 0.

For if one of the quantities is not one-valued, and none of the others is the number 0, we can choose a non-nul value of each of these, which when multiplied together with different values of the first quantity, necessarily gives rise to different values of the product.

The *distributive law for addition and multiplication* takes the form

$$(a+b)c \subset ac + bc,$$

hence more generally

$$(a+b)(c+d) \subset ac + bc + ad + bd.$$

If c is one-valued, or if a or b is 0, the former relation of inclusion obviously reduces to an equality. In *other cases* it may or may not reduce to equality. Thus if c has the two values 0 and 1, those of $(a+b)c$ are the values of $(a+b)$ and the value 0, while those of $(ac+bc)$ are those of a, of b, of $(a+b)$, and 0; but in this case we have certainly

$$(a+b)c = ac + bc$$

if both a and b include the value 0; since

$$a \subset a + b$$

if b includes the value 0, and

$$b \subset a + b$$

if a includes the value 0.

Note. As a particular corollary of this proposition, we note that for every a:

$$a_+ + a_- \supset a; \qquad a_+ - a_- \supset |a|.$$

These relations are obvious, and reduce indeed to equalities, if a is positive or negative. In every other case, both a_+ and a_- include the value 0, and hence are included in their sum; and both a_+ and $-a_-$ are included in theirs. Since every value of a is one of a_+ or of a_-, and every value of $|a|$ is one of a_+ or of $-a_-$, the truth of the two relations follows.

For the *associated quantities of sums and products* we have the following rules:

Upper and lower values[5]):

$$\overline{a+b} = \bar{a} + \bar{b}, \qquad \underline{a+b} = \underline{a} + \underline{b};$$

$$\overline{a \cdot b} = \bar{a} \cdot \bar{b}, \qquad \underline{a \cdot b} = \underline{a} \cdot \underline{b} \quad \text{if } a, b \text{ are positive}$$

and generally

$$\left.\begin{array}{l} \overline{a \cdot b} = \text{largest} \\ \underline{a \cdot b} = \text{least} \end{array}\right\} \text{ of } \bar{a}\cdot\bar{b},\ \bar{a}\cdot\underline{b},\ \underline{a}\cdot\bar{b},\ \underline{a}\cdot\underline{b}.$$

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[5]) Subject to existence (finite).

As particular useful deductions, we note

$$\overline{a-a}=\beta_a, \qquad \underline{a-a}=-\beta_a;$$
$$\beta_{a+b}=\beta_a+\beta_b;$$
$$\beta_{(ab)}\geqq\beta_{(|a|\cdot|b|)}\geqq|a|\cdot\beta_b.$$

Absolute magnitude.

$$|a+b|\leqq|a|+|b|; \quad |a-b|\geqq\big||a|-|b|\big|; \quad |ab|=|a|\cdot|b|.$$

There are formally precisely the same rules as for one-valued quantities and follow from these.

$+$ and $-$ parts.

$$(a+b)_+\leqq a_++b_+;$$
$$(a+b)_-\geqq a_-+b_-;$$

(*equality occurs when a and b are both positive, or both negative*).

The second relation obviously reduces to the first when we substitute $-a$ for a and $-b$ for b in it.

To prove the first, we note that a value of

$$(a+b)_+$$

is either 1. the sum of a positive value of a and a positive value of b, i. e. a value of (a_++b_+); or 2. it is the sum of a positive value of a and a non-positive one of b, i. e. is $\leqq$ a value of a_+; or 3. the sum of a non-positive value of a and a positive one of b, i. e. is $\leqq$ a value of b_+; or 4. it is 0 with $(a_++b_+)>0$. In each of these cases it is a fortiori $\leqq$ a value of (a_++b_+). Thus each value of the left-hand side is actually $\leqq$ some value of the right-hand side.

Conversely, a value of

$$a_++b_+$$

is either 1. the sum of a positive value of a and a positive value of b, hence a positive value of $(a+b)_+$; or 2. it is a positive value of a, and b includes some value $\leqq 0$, hence $(a+b)$, and so $(a+b)_+$, include some value $\leqq$ that positive value of a; or 3. it is a positive value of b, and $(a+b)_+$ includes similarly some value $\leqq$ it; or 4. it is 0, and both a and b include non-positive values, hence also $(a+b)_+>0$. In each of these cases, some value of $(a+b)_+$ is $\leqq$ the assumed value of a_++b_+. And each value of the right-hand side is thus also some value of the left. Q. e. d.

Frame.

$$\boxed{a+b}=\boxed{a}+\boxed{b};$$
$$\boxed{ab}=\boxed{a}\cdot\boxed{b}.$$

The definitions are extended to $\bar{o}$ by writing

$$a+\bar{o}=a-\bar{o}=a; \qquad a\cdot\bar{o}=\frac{a}{\bar{o}}=a;$$

and generally following the principle that *in all calculations with sums, differences, products and ratios*, $\bar{o}$ has no effect, and may be removed, introduced and transfered at will.

5. New operations.

We define the *union*

$$a\smile b$$

of two quantities a and b as the quantity having all, and only, the values of a and the values of b.

We define the *link*

$$a\between b$$

of two quantities a and b as the quantity having all, and only, the values common to both a and b. If such values do not exist, the link of a and b is nought.

The definitions are extended to nought by

$$(a\smile\bar{o})=a,$$
$$a\between\bar{o}=\bar{o},$$

We have

$$(a\smile a)=(a\between a)=a;$$
$$(a\between b)\subset a\subset(a\smile b).$$

The definitions are immediately extended to more than two quantities a and b, so that we may speak of the *union* and the *link of any number of many-valued quantities.*

The new operations are obviously *commutative* and *associative* in the ordinary sense. Moreover their combination with one another is *distributive* in the ordinary sense, i. e.

$$(a\smile b)\between c=(a\between c)\smile(b\between c).$$

As regards their combination with former operations, we have

$$(a\smile b)+c=(a+c)\smile(b+c),$$
$$(a\smile b)\cdot c=(ac)\smile(bc);$$
$$(a\between b)+c\subset(a+c)\between(b+c),$$
$$(a\between b)\cdot c\subset(ac)\between(bc);$$
$$(a+b)\smile(c+d)\subset(a\smile c)+(b\smile d),$$
$$(ab)\smile(cd)\subset(a\smile c)\cdot(b\smile d);$$
$$(a+b)\between(c+d)\supset(a\between c)+(b\between d).$$
$$(ab)\between(cd)\supset(a\between c)\cdot(b\between c).$$

We have also obviously the *inclusion laws*[6]): if $a \subset b$, then, for every c,

$$(a \cup c) \subset (b \cup c), \quad (a \mathbin{\ell} c) \subset (b \mathbin{\ell} c).$$

As regards the associated quantities of links and unions, we have the rules:

Upper and lower values.

$$\overline{a \cup b} = \text{larger of} \begin{Bmatrix} \bar{a} \\ \text{and} \\ \bar{b} \end{Bmatrix} \geqq \bar{a} \cup \bar{b},$$

$$\underline{a \cup b} = \text{lesser of} \begin{Bmatrix} \underline{a} \\ \text{and} \\ \underline{b} \end{Bmatrix} \leqq \underline{a} \cup \underline{b};$$

$$\overline{a \mathbin{\ell} b} \leqq \bar{a} \cup \bar{b},$$

$$\underline{a \mathbin{\ell} b} \geqq \underline{a} \cup \underline{b}.$$

Absolute magnitude.

$$|a \cup b| = |a| \cup |b|,$$

$$|a \mathbin{\ell} b| \subset |a| \mathbin{\ell} |b|.$$

Opposite.

$$-(a \cup b) = (-a) \cup (-b), \quad -(a \mathbin{\ell} b) = (-a) \mathbin{\ell} (-b).$$

Inverse.

$$\frac{1}{a \cup b} = \frac{1}{a} \cup \frac{1}{b}; \quad \frac{1}{a \mathbin{\ell} b} = \frac{1}{a} \mathbin{\ell} \frac{1}{b}.$$

$+$ and $-$ parts.

$$(a \cup b)_+ = a_+ \cup b_+; \quad (a \cup b)_- = a_- \cup b_-.$$

Frame.

$$\boxed{a \cup b} = \boxed{a} \cup \boxed{b}; \quad \boxed{a \mathbin{\ell} b} \subset \boxed{a} \mathbin{\ell} \boxed{b}.$$

6. Special many-valued quantities.

The *symmetrical sub-unit*

$$\theta$$

of many-valued quantities is the quantity having all, and only, the values between -1 and 1 both inclusive.

We have clearly

$$\theta = -\theta \quad \text{(symmetry)},$$

$$\theta \cdot \theta = \theta$$

and

$$\theta + \theta = 2\theta.$$

[6]) As regards the *monotony law*, this only holds for a union, in the form: if $a \leqq b$, then, for every c,

$$(a \cup c) \leqq (b \cup c).$$

If δ is any number, the product

$$\delta\theta$$

has all, and only, the values between $\pm\delta$ inclusive, thus representing in fact the closed interval of endpoints $-\delta$, δ on the number-axis.

For any many-valued a, the sum

$$a + \delta\theta$$

has all, and only, the values each of which differs from some value of a by not more than $|\delta|$. It includes in particular all the values and all the limiting values of a (the latter provided δ is not 0); i. e. *for every positive number* δ,

$$a + \delta\theta \supset \boxed{a}.$$

Moreover, we have

$$a + \delta\theta \subset \boxed{a} + \delta\theta.$$

An *infinitesimal*

$$e$$

is any positive many-valued quantity whose lower value is 0. The letter

$$\varepsilon$$

will represent *any* value of such an infinitesimal, and is thus, in the usual language of analysis, an "*arbitrarily small number*". Thus also the expression

$$\varepsilon\theta$$

represents an "arbitrarily small" interval of centre 0, and

$$a + \varepsilon\theta$$

an "arbitrarily close neighbourhood" of the set of values of a.

Accordingly we see at once that

$$c \subset \boxed{a}$$

if, and only if,

$$c \subset a + \varepsilon\theta$$

(*for every value* ε *of an infinitesimal*).

For any given a, if K be a sufficiently large positive number, the link

$$a \mathbin{\ell} K\theta$$

of a and $K\theta$ certainly has a value. We denote it also by

$$\rfloor a \rfloor_K.$$

The process itself, namely of forming the link of a given many-valued quantity and of the special $K\theta$, will be described as *levelling* the given quantity.

By the distributive law for links of sums,

$$\overline{a}\big]_K + \overline{b}\big]_K \prec \overline{(a+b)}\big]_{2K}.$$

A kind of converse is provided by the following useful property:

If a horizontal or vertical pair[7]) *of the four relations*

$$\begin{array}{ll} a < K, & b < K, \\ -a < K, & -b < K, \end{array}$$

is known to hold, then

$$\overline{(a+b)}\big]_K \prec \overline{a}\big]_{2K} + \overline{b}\big]_{2K}.$$

By interchanging a and $-b$, we can always reduce any of the supposed pairs to include

$$a < K.$$

As the required relation is then transformed into itself (by taking opposites), it suffices to consider this case. It then stands in conjunction with either

$$-a < K \quad \text{or} \quad b < K.$$

Now every value β of b which, with some value α of a, gives a sum included in $K\theta$, so that in particular

$$\alpha + \beta \geqq -K,$$

must satisfy

$$\beta \geqq -2K,$$

since by hypothesis α is $< K$.

If $-a < K$, we have similarly (or by writing $-a$ for a, $-b$ for b)

$$\beta \leqq 2K.$$

Hence both in this case and in the case $b < K$, we have necessarily

$$\beta \prec 2K\theta.$$

In the former case, as we have already

$$a \prec K\theta \prec 2K\theta,$$

this proves the required relation. In the latter case, in which the hypotheses are symmetrical in a and b, we see by interchanging a and b that

7) I. e. either a and b are both $< K$ or are $< -K$, or one of them at least between $\pm K$.

also every value α of a which, with some value β of b, gives a sum included in $K\theta$, must necessarily satisfy

$$\alpha \prec 2K\theta,$$

and the required relation is again established. This completes the proof.

By the distributive law for links of products,

$$\overline{a}\big]_K \cdot \overline{b}\big]_K \prec \overline{(a \cdot b)}\big]_{K^2}.$$

A kind of converse is again provided by the proposition:

If a vertical or horizontal pair of the relations

$$\begin{array}{ll} a \prec K\theta, & b \prec K\theta, \\ \frac{1}{a} \prec K\theta, & \frac{1}{b} \prec K\theta, \end{array}$$

is known to hold, then

$$\overline{(a \cdot b)}\big]_K \prec \overline{a}\big]_{K^2} \cdot \overline{b}\big]_{K^2}.$$

The proof runs exactly parallel to the preceding one. Since interchanging a and $\frac{1}{b}$, and taking inverses, simply transforms the required relation into itself, we may suppose

$$\frac{1}{a} \prec K\theta$$

to be one of the assumed relations. In that case, every value β of b for which, with some value α of a,

$$\alpha \cdot \beta \prec K\theta$$

must satisfy

$$\beta \prec K^2\theta.$$

And either directly, if $a \prec K\theta$ is the other assumed relation, or by interchanging a and b if $\frac{1}{b} \prec K\theta$ is the other assumed relation, we must also have

$$\alpha \prec K^2\theta.$$

Hence q. e. d.

7. Limits.

Our mode of extending to "many-valued" quantities the operations of ordinary algebra applies, mutatis mutandis, to the process of *passage to the limit.*

A. Successions.

A succession

$$a_1, a_2, \ldots, a_m, \ldots$$

of quantities is said to have as its *finite limit*, denoted by

$$\left(\lim_{m\to\infty}\right) a_m,$$

the quantity a having all, and only, the values which are finite numerical (upper, lower or intermediary) limits, in the ordinary sense, of successions whose m^{th} term is a value of a_m. This definition is quite unambiguous and always yields a quantity a, which may, however, in particular cases, reduce to $\bar{o}$.

If every succession whose m^{th} term is a value of a_m is bounded (i. e. no such succession has an infinite upper or lower limit in the ordinary sense), the finite limit of the succession

$$a_1, a_2, \ldots, a_m, \ldots$$

is called its *complete limit*, and is denoted by

$$\lim_{m\to\infty} a_m.$$

A complete limit cannot be nought, and is always strictly finite.

Theorem I. *A value α belongs to*

$$\left(\lim_{a\to\infty}\right) a_r$$

if, and only if, it belongs to

$$a_m + \theta\varepsilon$$

for a sequence of indices

$$m = m_i(\alpha, \varepsilon).$$

This follows at once from the definition of a numerical limit in the ordinary sense. It may also be taken as the definition of the finite limit, and then includes as a particular case that of a unique numerical limit in the ordinary sense.

Theorem II. *If a is the complete limit of a succession of quantities a_m, then*

$$a_m < a + \theta\varepsilon \quad \text{for all} \quad m > N_\varepsilon.$$

For a value α_m of a_m not included in $a + \theta\varepsilon$ is one differing by more than ε from every value of a. If such a value exists for a sequence of indices m_i, each numerical limit of the succession $\{\alpha_{m_i}\}$ differs from every value of a (by not less than ε), whereas since a is the complete limit of a_m, it must exist (finite) and belong to a by definition.

From Theorem I, we may at once deduce that

The finite limit of a succession of quantities is always closed finitely. A complete limit is therefore completely closed.

Let a be the finite limit of a succession

$$a_1, a_2, \ldots, a_m, \ldots$$

and let α be any limiting value of a. The statement is that α is a value of a.

By its definition as a limiting value of a, the number α belongs to

$$\left(\lim_{m\to\infty}\right) \alpha_m$$

where the numbers α_m are values of a. Hence by Theorem I,

$$\alpha < \alpha_k + \theta\varepsilon/2$$

for a sequence of indices

$$k = k_i(\alpha, \varepsilon).$$

Also, as values of $\left(\lim_{m\to\infty}\right) a_m$, the numbers α_k, again by Theorem I, belong to

$$a_m + \theta\varepsilon/2$$

each for a sequence of indices

$$m = m_j(\alpha_k, \varepsilon).$$

Hence, for this sequence of indices m,

$$\alpha_k + \theta\varepsilon/2 < a_m + \theta\varepsilon.$$

It follows that

$$\alpha < a_m + \theta\varepsilon$$

for the double sequence of indices $m = m_{ij}(\alpha, \varepsilon) = m_j(\alpha_{k_i(\alpha,\varepsilon)}, \varepsilon)$, and a fortiori the condition of Theorem I is fulfilled for α. Hence q. e. d.

The following properties are immediate:

$$\text{If} \quad a_1 = a_2 = \ldots = a_m = \ldots = a, \quad \left(\lim_{m\to\infty}\right) a_m = \boxed{a}.$$

$$\text{If} \quad a_m \leqq b_m \quad \text{for each } m, \quad \left(\lim_{m\to\infty}\right) a_m \leqq \left(\lim_{m\to\infty}\right) b_m.$$

$$\text{If} \quad a_m > b_m \quad \text{for each } m, \quad \left(\lim_{m\to\infty}\right) a_m > \left(\lim_{m\to\infty}\right) b_m.$$

Also

$$\left(\lim_{m\to\infty}\right) a_{m+p} = \left(\lim_{m\to\infty}\right) a_m.$$

As a particular case of the third property, obtained from it by taking $b = \bar{o}$ for all indices except $m_1, m_2, \ldots$, we may state:

For every sequence of indices m_i,

$$\left(\lim_{i\to\infty}\right) a_{m_i} < \left(\lim_{m\to\infty}\right) a_m.$$

As a consequence of the fourth of the above properties, we may also speak of the finite limit of a succession whose first p terms are not all properly defined, writing

$$\left(\lim_{m\to\infty}\right) a_m = \left(\lim_{m\to\infty}\right) a_{m+p}$$

then as the *definition* of the left-hand side. This is convenient e. g. when taking inverses (see below).

The upper and lower values of $\left(\lim\limits_{m\to\infty}\right) a_m$, when defined (finite) will be denoted for simplicity by

$$\overline{\left(\lim_{m\to\infty}\right)}\, a_m, \quad \underline{\left(\lim_{m\to\infty}\right)}\, a_m$$

and called the *upper and lower finite limits* of a_m $(m \to \infty)$. If the limit is complete, both are of course defined, and we then denote them by

$$\overline{\lim_{m\to\infty}}\, a_m, \quad \underline{\lim_{m\to\infty}}\, a_m,$$

and call them the *upper and lower limits.* We have obviously, in the latter case,

$$\overline{\lim_{m\to\infty}}\, a_m = \overline{\lim_{m\to\infty}}\, \bar{a}_m, \quad \underline{\lim_{m\to\infty}}\, a_m = \underline{\lim_{m\to\infty}}\, \underline{a}_m,$$

and in general *one or other of*

$$\overline{\left(\lim_{m\to\infty}\right)}\, a_m = \overline{\left(\lim_{m\to\infty}\right)}\, \bar{a}_m, \quad \underline{\left(\lim_{m\to\infty}\right)}\, a_m = \underline{\left(\lim_{m\to\infty}\right)}\, \underline{a}_m$$

holds provided only the right-hand side of it is defined. It is at once obvious that if (and only if) *both* right-hand sides are defined, the limit is complete.

Again, we have

$$\left|\left(\lim_{m\to\infty}\right) a_m\right| = \left(\lim_{m\to\infty}\right) |a_m|; \quad -\left(\lim_{m\to\infty}\right) a_m = \left(\lim_{m\to\infty}\right)(-a_m).$$

And

$$\left(\left(\lim_{m\to\infty}\right) a_m\right)_+ = \left(\lim_{m\to\infty}\right)(a_m)_+; \quad \left(\left(\lim_{m\to\infty}\right) a_m\right)_- = \left(\lim_{m\to\infty}\right)(a_m)_-.$$

To see this, we have only to remark firstly that α is a positive value of $\left(\lim\limits_{m\to\infty}\right) a_m$ if, and only if, it is a limit of a succession of positive values α_{m_i} of a_{m_i}; so that the positive values of

$$\left(\left(\lim_{m\to\infty}\right) a_m\right)_+ \quad \text{and} \quad \left(\lim_{m\to\infty}\right)(a_m)_+$$

are certainly the same; next as regards the value 0,

I) if a value of $\left(\lim\limits_{m\to\infty}\right) a_m$ as a limit of positive values α_{m_i}, it again belongs to both;

II) if a value of $\left(\lim\limits_{m\to\infty}\right) a_m$ as a limit of negative values α_{m_i}, it belongs to $(a_{m_i})_+$ and hence to $\left(\lim\limits_{m\to\infty}\right)(a_m)_+$, while it also belongs to $\left(\left(\lim\limits_{m\to\infty}\right) a_m\right)_+$;

III) if a value of $\left(\left(\lim\limits_{m\to\infty}\right) a_m\right)_+$ but not of $\left(\lim\limits_{m\to\infty}\right) a_m$, then the latter has negative values, and hence so has a_m for a sequence of indices m_i, so that again $(a_{m_i})_+$ includes 0, and so does $\left(\lim\limits_{m\to\infty}\right)(a_m)_+$.

This completes the proof of the first equality; the second follows by writing $-a$ for a, by the relation for opposites just before.

We have also, as an immediate deduction from Theorem I,

$$\left(\lim_{m\to\infty}\right) a_m = \boxed{\left(\lim_{m\to\infty}\right) a_m} = \left(\lim_{m\to\infty}\right) \boxed{a_m},$$

since $\boxed{a_m} \subset a_m + \theta\varepsilon \subset \boxed{a_m} + \theta\varepsilon$.

Finally,

$$\frac{1}{\left(\lim\limits_{m\to\infty}\right) a_m} = \left(\lim_{m\to\infty}\right) \frac{1}{a_m}$$

provided only the left-hand side is defined.

B. Double successions.

A double succession

$$\begin{array}{l} a_{11}, a_{12}, \ldots, a_{1m}, \ldots \\ a_{21}, a_{22}, \ldots, a_{2m}, \ldots \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{k1}, a_{k2}, \ldots, a_{km}, \ldots \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}$$

of quantities is said to have as its *finite double limit*

$$\left(\lim_{(k,m)\to\infty}\right) a_{k,m},$$

the quantity a having all, and only, the values which are finite numerical (upper, lower, or intermediary) *double limits* in the ordinary sense, of double successions having as term of index (k, m) a value of $a_{k,m}$. When all such double successions are bounded, the finite double limit is also called the *complete double limit,* and denoted by

$$\lim_{(k,m)\to\infty} a_{k,m}.$$

If, in this definition, we take only one type of *repeated* (and not all the double) numerical limits, we obtain a quantity included in the finite double limit which we call a *finite repeated limit* of the double succession. The twin type of repeated numerical limits then give another finite repeated limit of the double succession. It is at once clear that the two repeated limits may be obtained as

$$\left(\lim_{k\to\infty}\right)\left(\left(\lim_{m\to\infty}\right) a_{k,m}\right) \quad \text{and} \quad \left(\lim_{m\to\infty}\right)\left(\left(\lim_{k\to\infty}\right) a_{k,m}\right)$$

respectively, and we denote them by

$$\left(\lim_{k\to\infty}\lim_{m\to\infty}\right) a_{k,m}, \quad \left(\lim_{m\to\infty}\lim_{k\to\infty}\right) a_{k,m}$$

respectively; omitting the brackets when the limits are complete.

We have, as remarked,

$$\left(\lim_{k\to\infty}\lim_{m\to\infty}\right)a_{k,n} \smile \left(\lim_{m\to\infty}\lim_{k\to\infty}\right)a_{k,m} \subset \left(\lim_{(k,m)\to\infty}\right)a_{k,m}.$$

We have the analogues of Theorems I and II:

Theorem III. *A value α belongs to*

$$\left(\lim_{(m,n)\to\infty}\right)a_{m,n}$$

if, and only if, it belongs to

$$a_{m,n}+\theta\varepsilon$$

for a sequence of indices

$$m=m_i(\alpha,\varepsilon),\qquad n=n_i(\alpha,\varepsilon)$$

(*both* tending of course to ∞ with i).

Theorem IV. *If a is the complete double limit of a double succession of quantities $a_{m,n}$, then*

$$a_{m,n}\subset a+\theta\varepsilon \quad \textit{for all}\quad m>N_\varepsilon \quad \textit{and all}\quad n>N_\varepsilon.$$

All the other properties of the finite limit of a succession hold in the exactly parallel form for finite double limits of double successions, and we use the corresponding notation

$$\overline{\left(\lim_{(m,n)\to\infty}\right)}a_{m,n} \quad\text{and}\quad \underline{\left(\lim_{(m,n)\to\infty}\right)}a_{m,n}$$

to denote the upper and lower values of the finite double limit, or *upper* and *lower finite double limits*, when existent (finite), omitting the brackets when dealing with a *complete* limit.

C. *Nple successions.*

In like manner, we define and discuss the *finite Nple limit* of an Nple succession of quantities

$$a_{(m)}$$

where (m) stands for N indices $m_1, m_2, \ldots, m_n$, each of which assumes all integral values. We denote the finite Nple limit by

$$\left(\lim_{(m)\to\infty}\right)a_{(m)}$$

omitting the brackets when the limit is *complete*; call its upper and lower values, as far as defined, the *upper* and *lower finite Nple limits* and denote them by

$$\overline{\left(\lim_{(m)\to\infty}\right)}a_{(m)},\qquad \underline{\left(\lim_{(m)\to\infty}\right)}a_{(m)};$$

and have all the parallel properties holding indiscriminately. We have also the various *repeated limits*, all included in the double one,

$$\left(\lim_{m_1'\to\infty}\lim_{m_2'\to\infty}\cdots\lim_{m_N'\to\infty}\right)a_{(m)},$$

where $m_1', m_2', \ldots, m_N'$ is any permutation of the indices $(m_1, m_2, \ldots, m_N)=(m)$.

And we have *partially repeated limits*

$$\left(\lim_{(m)_1\to\infty}\lim_{(m)_2\to\infty}\cdots\lim_{(m)_k\to\infty}\right)a_{(m)},$$

where $(m)_1, (m)_2, \ldots, (m)_k$ represent mutually exclusive groups of the indices (m), together comprising all the indices (m).

8. Distributive laws for limits.

Denoting as agreed by

$$((m),(n))$$

the aggregate of all indices

$$m_1, m_2, \ldots, m_M, n_1, n_2, \ldots, n_N,$$

of

$$(m)=(m_1, m_2, \ldots, m_M)$$

and

$$(n)=(n_1, n_2, \ldots, n_N),$$

we have obviously (provided, in the first two relations, neither of the finite limits on the left hand side is $\bar{o}$; but without restriction in the last):

$$\text{(X)}\quad\begin{cases}\left(\lim_{(m)\to\infty}\right)a_{(m)}+\left(\lim_{(n)\to\infty}\right)b_{(n)}=\left(\lim_{(m)\to\infty}\lim_{(n)\to\infty}\right)\left(a_{(m)}+b_{(n)}\right),\\ \left(\lim_{(m)\to\infty}\right)a_{(m)}\cdot\left(\lim_{(n)\to\infty}\right)b_{(n)}=\left(\lim_{(m)\to\infty}\lim_{(n)\to\infty}\right)\left(a_{(m)}\cdot b_{(n)}\right),\\ \left(\lim_{(m)\to\infty}\right)a_{(m)}\smile\left(\lim_{(n)\to\infty}\right)b_{(n)}=\left(\lim_{(m)\to\infty}\lim_{(n)\to\infty}\right)\left(a_{(m)}\smile b_{(n)}\right).\end{cases}$$

These relations follow at once from the fact that a finite limit is closed finitely, i. e. coincides with its frame, and the fact that the finite limit of a succession of terms all equal to a is $\boxed{a}$, (or the parallel facts for Nple successions).

As all the operations are commutative, the limits on the right-hand sides may also be replaced by those of the twin type

$$\left(\lim_{(n)\to\infty}\lim_{(m)\to\infty}\right),$$

so that in these cases, i. e. for $(M+N)$ple successions whose term of index $((m),(n))$ is always either the sum, or the product, or the union, of the term $a_{(m)}$ of a Mple succession and the term $b_{(n)}$ of a Nple succession, the two finite partially repeated limits

$$\left(\lim_{(m)\to\infty}\lim_{(n)\to\infty}\right),\qquad \left(\lim_{(n)\to\infty}\lim_{(m)\to\infty}\right)$$

are equal.

The corresponding formula for links is

$$\Big(\lim_{(m)\to\infty}\ \lim_{(n)\to\infty}\Big)(a_{(m)} \,\&\, b_{(n)}) \subset \Big(\lim_{(m)\to\infty}\Big)a_{(m)} \,\&\, \Big(\lim_{(n)\to\infty}\Big)b_{(n)}$$

$$\subset \Big(\lim_{(m)\to\infty}\ \lim_{(n)\to\infty}\Big)(a_{(m)} \,\&\, b_{(n)}) \cup \Big\{\Big(\lim_{(m)\to\infty}\Big)c_m \,\&\, \Big(\lim_{(n)\to\infty}\Big)b_{(n)}\Big\},$$

where $c_{(m)}$ has all and only the values of $a_{(m)}$ not belonging to $\Big(\lim_{(n)\to\infty}\Big)b_{(n)}$. It is because the latter link is not always $\dot{o}$ that we do not in general get equality.

In the case of unions and links, we can at once complete the result, by shewing that

$$\Big(\lim_{((m),(n))\to\infty}\Big)(a_{(m)} \cup b_{(n)}) = \Big(\lim_{(m)\to\infty}\Big)a_{(m)} \cup \Big(\lim_{(n)\to\infty}\Big)b_{(n)},$$

$$\Big(\lim_{((m),(n))\to\infty}\Big)(a_{(m)} \,\&\, b_{(n)}) \subset \Big(\lim_{(m)\to\infty}\Big)a_{(m)} \,\&\, \Big(\lim_{(n)\to\infty}\Big)b_{(n)}.$$

These relations are simply expressions of the fact that a succession of numbers taken from two given successions of numbers has for its numerical limits exclusively limits of these two successions; and if each number of the first succession belongs to *both* the given successions, each of its limits is a limit of both given successions. These facts are equivalent, indeed to the statement that the left-hand sides of the above equalities are included in the right. The converse, in the case of unions, we already know by (X).

We note that

$$\Big(\lim_{m\to\infty}\Big)\overline{a}|_m = \boxed{a}.$$

This is obtained by writing

$$a_m = a, \quad b_n = n\theta$$

in the general formula for links, and observing that

$$\Big(\lim_{n\to\infty}\Big)n\theta$$

has all possible values, so that its link with any quantity is again that quantity. Another particular case to note is

$$\boxed{\Big(\lim_{(m)\to\infty}\Big)a_{(m)}}_K = \lim_{(m)\to\infty}\boxed{a_{(m)}}_K,$$

except for possible values $\pm K$ of the left-hand side, not necessarily belonging to the right.

The case of sums and products is however more subtle. This is at once clear if we think of a succession of quantities a_m whose finite limit is $\dot{o}$ (as for instance when $a_m = m$), and take $b_m = -a_m$. The sum $a_m + b_n$ then includes the value 0 for every $m = n$, and hence its finite double limit includes the value 0, whereas $\Big(\lim_{m\to\infty}\Big)a_m + \Big(\lim_{n\to\infty}\Big)b_n$ is $\dot{o}$. Similarly, if, for the same a_m, we take $b_m = \dfrac{1}{a_m}$, the product $a_m \cdot b_n$ includes

the value 1 for every $m = n$, and hence so does its finite double limit, whereas $\Big(\lim_{m\to\infty}\Big)a_m \cdot \Big(\lim_{n\to\infty}\Big)b_n = \dot{o}\cdot 0 = \dot{o}$.[8])

We may however at once prove the

Theorem V. *For any Mple and any Nple succession, of terms $a_{(m)}$ and $b_{(n)}$ respectively, whose limits are complete,*

$$\lim_{((m),(n))\to\infty}(a_{(m)} + b_{(n)}) = \lim_{(m)\to\infty}a_{(m)} + \lim_{(n)\to\infty}b_{(n)},$$

$$\lim_{((m),(n))\to\infty}(a_{(m)} \cdot b_{(n)}) = \lim_{(m)\to\infty}a_{(m)} \cdot \lim_{(n)\to\infty}b_{(n)}.$$

This is a consequence of Theorem II for successions with complete finite limits, if we confine ourselves to the case $M = N = 1$ (the proof being precisely parallel in the general case). In fact, by this theorem, we certainly have for all $m > N_\varepsilon$, and all $n > N_\varepsilon$,

$$\text{(x)} \qquad a_m \subset \lim_{m\to\infty} a_m + \theta\varepsilon, \quad b_n \subset \lim_{n\to\infty} b_n + \theta\varepsilon;$$

hence by addition and passage to the limit,

$$\lim_{(m,n)\to\infty}(a_m + b_n) \subset \lim_{m\to\infty} a_m + \lim_{n\to\infty} b_n + 2\theta\varepsilon,$$

since both limits on the right-hand side are closed; and for this same reason, the latter relation is equivalent (cp. p. 273) to

$$\lim_{(m,n)\to\infty}(a_m + b_n) \subset \lim_{m\to\infty} a_m + \lim_{n\to\infty} b_n,$$

which is the addition form of the theorem.

Similarly, by multiplication and passage to the limit from (x)

$$\lim_{(m,n)\to\infty}(a_m \cdot b_n) \subset \lim_{m\to\infty} a_m \cdot \lim_{n\to\infty} b_n + A\theta\varepsilon,$$

where

$$A = \lim_{m\to\infty} a_m + \lim_{n\to\infty} b_n + \theta\varepsilon$$

---

[8]) Here the complication is not, — as in the corresponding two relations (X), — removed by merely stipulating that neither $\Big(\lim_{m\to\infty}\Big)a_m$ nor $\Big(\lim_{n\to\infty}\Big)b_n$ is $\dot{o}$.

For instance, if, in the two examples of the text, we replace

$$a_m \text{ by } (a_m \cup c)$$
$$b_n \text{ by } (b_n \cup d)$$

for each m, n then by the relation (X) for unions, and the distributive laws for the union of sums or products, the finite limits on the one side are in both cases $\boxed{c}$ and $\boxed{d}$, while those on the other include

$$0 \cup \boxed{c+d} \quad \text{and} \quad 1 \cup \boxed{c\cdot d}$$

respectively, — which by choice of c and d can easily be made to differ from $\boxed{c+d}$ and $\boxed{c\cdot d}$.

is strictly finite, and so Ae is always an infinitesimal. This is thus again equivalent to

$$\lim_{(m,n)\to\infty}(a_m\cdot b_n)=\lim_{m\to\infty}a_m\cdot\lim_{n\to\infty}b_n,$$

which is the product form of the theorem.

To generalise Theorem V as far as possible, we use the process of *levelling*, and its properties noted on pp. 274, 275.

Given any Mple and any Nple succession, of terms $a_{(m)}$ and $b_{(n)}$ respectively, those of terms

$$\boxed{a_{(m)}}_K,\quad \boxed{b_{(n)}}_K$$

respectively have *complete* limits, and hence, by the theorem,

$$(\mathrm{y})\quad\begin{cases}\lim\limits_{((m),(n))\to\infty}\left\{\boxed{a_{(m)}}_K+\boxed{b_{(n)}}_K\right\}=\lim\limits_{(m)\to\infty}\boxed{a_{(m)}}_K+\lim\limits_{(n)\to\infty}\boxed{b_{(n)}}_K,\\ \lim\limits_{((m),(n))\to\infty}\left\{\boxed{a_{(m)}}_K\cdot\boxed{b_{(n)}}_K\right\}=\lim\limits_{(m)\to\infty}\boxed{a_{(m)}}_K\cdot\lim\limits_{(n)\to\infty}\boxed{b_{(n)}}_K.\end{cases}$$

The right-hand sides, which are included in

$$\boxed{\left(\lim_{(m)\to\infty}\right)a_{(m)}}_K+\boxed{\left(\lim_{(n)\to\infty}\right)b_{(n)}}_K,\qquad \boxed{\left(\lim_{(m)\to\infty}\right)a_{(m)}}_K\cdot\boxed{\left(\lim_{(n)\to\infty}\right)b_{(n)}}_K$$

respectively, are therefore included in

$$(\mathrm{z})\qquad \left(\lim_{(m)\to\infty}\right)a_{(m)}+\left(\lim_{(n)\to\infty}\right)b_{(n)},\qquad \left(\lim_{(m)\to\infty}\right)a_{(m)}\cdot\left(\lim_{(n)\to\infty}\right)b_{(n)}$$

respectively. And *under the conditions of p. 274 and p. 275 respectively* the two { } brackets on the left in (y) include

$$\boxed{(a_{(m)}+b_{(n)})}_{2K},\qquad \boxed{(a_{(m)}\cdot b_{(n)})}_{K^2}$$

respectively, and so the two limits on the left in (y) include, except for possible values $\pm 2K$, $\pm K^2$,

$$\boxed{\left(\lim_{((m),(n))\to\infty}\right)(a_{(m)}+b_{(n)})}_{2K},\qquad \boxed{\left(\lim_{((m),(n))\to\infty}\right)(a_{(m)}\cdot b_{(n)})}_{K^2}$$

respectively; by passage to the limit for $K\to\infty$, these become

$$\left(\lim_{((m),(n))\to\infty}\right)(a_{(m)}+b_{(n)}),\qquad \left(\lim_{((m),(n))\to\infty}\right)(a_{(m)}\cdot b_{(n)}).$$

Now the assumed conditions, if they hold for some K, hold a fortiori for all larger K. Hence we deduce that the latter limits are included in, and hence, by (X), equal to, the corresponding expressions (z).

The assumed conditions are, respectively, that a horizontal or vertical pair of the four relations

$$a_{(m)}<K,\qquad b_{(n)}<K,$$
$$-a_{(m)}<K,\qquad -b_{(n)}<K,$$

(for the addition form) or of the four relations

$$a_{(m)}<K\theta;\qquad b_{(n)}<K\theta,$$
$$\frac{1}{a_{(m)}}<K\theta,\qquad \frac{1}{b_{(n)}}<K\theta,$$

(for the product form), — should be known to hold for each pair of indices (m), (n). This reduces to assuming that one the same such pair of relations should hold *for all* (m), (n); for the assumptions require that if any one relation does *not* hold for all values of the index, the diagonally opposite relation *should* hold for all values of the index; e. g. in the first case

$$a_{(m)}\nless K \text{ for some } (m) \text{ implies } -b_{(n)}<K \text{ for all } (n);$$
$$-a_{(m)}\nless K \text{ for some } (m) \text{ implies } b_{(n)}<K \text{ for all } (n);$$

so that the only possibilities are

$$-K<a_{(m)}<K \quad\text{for all } (m);$$

or $\quad a_{(m)}<K,\quad b_{(n)}<K$ for all (m), (n);

or $\quad -a_{(m)}<K,\quad -b_{(n)}<K$ for all (m), (n);

or $\quad -K<b_{(n)}<K$ for all (n).

Similarly in the second case.

As we are dealing with limits, which are unchanged when we neglect a finite number of the terms, it suffices to assume the respective conditions fulfilled *for all sufficiently large m and n*. Thus finally we obtain the following statements.

Theorem VI. *If a horizontal or vertical pair of the four limits*

$$\overline{\left(\lim_{(m)\to\infty}\right)}\bar a_{(m)},\qquad \overline{\left(\lim_{(n)\to\infty}\right)}\bar b_{(n)},$$
$$\underline{\left(\lim_{(m)\to\infty}\right)}\underline{a}_{(m)},\qquad \underline{\left(\lim_{(n)\to\infty}\right)}\underline{b}_{(n)},$$

exist (finite), then

$$\left(\lim_{((m),(n))\to\infty}\right)(a_{(m)}+b_{(n)})=\left(\lim_{(m)\to\infty}\right)a_{(m)}+\left(\lim_{(n)\to\infty}\right)b_{(n)}.$$

Theorem VII. *If a horizontal or vertical pair of the four limits*

$$\lim_{(m)\to\infty}a_{(m)},\qquad \lim_{(n)\to\infty}b_{(n)},$$
$$\lim_{(m)\to\infty}\frac{1}{a_{(m)}},\qquad \lim_{(n)\to\infty}\frac{1}{b_{(n)}},$$

exist complete, then

$$\left(\lim_{((m),(n))\to\infty}\right)(a_{(m)}\cdot b_{(n)})=\left(\lim_{(m)\to\infty}\right)a_{(m)}\cdot\left(\lim_{(n)\to\infty}\right)b_{(n)}.$$

This is the best possible statement we can get for the distributive laws for finite limits combined by addition or multiplication, the complete generalisation requiring in fact the theory of infinities.

9. General limiting processes.

With regard to the limits defined and discussed so far, it should be noted that the set of values of

$$\left(\lim_{(m)\to\infty}\right) a_{(m)}$$

is *not* the unique limiting set, in the sense of the theory of sets, of the set of values of $a_{(m)}$, nor even of that of $\boxed{a_{(m)}}$ [9]). In general, of course, these sets will have no unique limiting set, but only upper and lower limiting sets. It is easy to see that these are contained in the set of values of

$$\left(\lim_{(m)\to\infty}\right) a_{(m)},$$

but in general the latter will have further values not contained in the limiting sets.

There is, however one important case in which the set of values of $\left(\lim_{(m)\to\infty}\right) a_{(m)}$ may be identified with the limiting set of the set of values of $a_{(m)}$ as $(m)\to\infty$. This appears from the following Theorem:

If $a_{(m)}$ is always closed finitely, and

$$a_{(m)} \supset a_{(m')}$$

for every $(m') \gg (m)$ [10]), *then*

$$\left(\lim_{(m)\to\infty}\right) a_{(m)}$$

has all, and only, the values common to $a_{(m)}$ for every (m).

Any value α belonging to $a_{(m)}$ for every (m) is of the form

$$\lim_{(m)\to\infty} \alpha_{(m)} \quad \text{with} \quad \alpha_{(m)} \subset a_{(m)}$$

(namely for $\alpha_{(m)} = \alpha$), and belongs therefore certainly to

$$\left(\lim_{(m)\to\infty}\right) a_{(m)}.$$

On the other hand, from

$$a_{(m)} \supset a_{(m')} \quad \text{for all} \quad (m') \gg (m)$$

we deduce

$$\boxed{a_{(m)}} \supset \left(\lim_{(m)\to\infty}\right) a_{(m')},$$

[9]) This is a familiar distinction in ordinary analysis, where $\lim_{i\to\infty} \alpha_i$ is not the same thing as $\lim_{i\to\infty} (\alpha_i)$, where (α_i) represents the set having α_i as its only object. E. g. if $\alpha_i = 1 + \frac{1}{i}$, the first lim is 1, the second is the null-set and corresponds only to $\ddot{o}$.

[10]) $(m') \gg (m)$ means each index m'_i is $>$ the corresponding index m_i.

hence, as $a_{(m)}$ is closed finitely,

$$a_{(m)} \supset \left(\lim_{(m)\to\infty}\right) a_{(m)},$$

so that every value of the limit certainly belongs to $a_{(m)}$ for all (m).

This completes the proof.

Now for purposes of evaluation of limits, the general type of Nple succession may be reduced to the above special type (*closed contracting successions*) by virtue of the following property:

Given any Nple succession of quantities $a_{(m)}$, and defining

$$g_{(m)} = \underset{(m')\gg(m)}{U} (a_{(m')}) \;^{11})$$

to have all, and only, the values each of which belongs to $a_{(m')}$ for some $(m') \gg (m)$, we have

$$\left(\lim_{(m)\to\infty}\right) g_{(m)} = \left(\lim_{(m\to\infty)}\right) a_{(m)}.$$

For simplicity suppose $N = 1$.

As

$$g_m \supset a_m,$$

we certainly have

$$\left(\lim_{m\to\infty}\right) g_m \supset \left(\lim_{m\to\infty}\right) a_m.$$

For the converse, note that a value of the left-hand side is a limit of values γ_{m_i} of g_{m_i} for a sequence of indices m_i, i. e. of values $\alpha_{m'_i}$ of $a_{m'_i}$ for a sequence of indices $m'_i > m_i$, i. e. a value of $\left(\lim_{m\to\infty}\right) a_m$.

This completes the proof, in the case $N = 1$, and with the slight complication in the indices, the same proof is valid in general.

The Nple succession of quantities

$$\boxed{g_{(m)}}$$

is clearly of the required special type, for each term is closed and the term of index (m) includes those of index (m') for all $(m') \gg (m)$. And its finite limit coincides with

$$\left(\lim_{(m)\to\infty}\right) g_{(m)},$$

hence with

$$\left(\lim_{(m)\to\infty}\right) a_{(m)}.$$

[11]) By analogy with the notation Σa_m and Πa_m, we use $U(a_m)$ to denote the union of a finite, and by extension (in the obvious sense) of an enumerably infinite, set of quantities a_m.

Thus

Theorem VIII. *The finite limit of any Nple succession of quantities $a_{(m)}$ has all, and only, the values common to*

$$\boxed{g_{(m)}} = \boxed{\underset{(m')\geqq(m)}{U} a_{(m')}}$$

for all (m).

This characteristic property of the finite limit of an *N*ple succession is the one most convenient for purposes of generalisation.

Suppose for instance that we have to consider a quantity $a(\xi)$ defined as a (*many-valued*) *function* of a numerical variable ξ. In order to define

$$\lim_{\xi\to\xi_0} a(\xi),$$

we first form the function

$$g_{\xi_0}(\xi)$$

having all, and only, the values each of which belongs to

$$a(\xi')$$

for some ξ' in

$$|\xi'-\xi_0| \leqq |\xi-\xi_0|,$$

i. e. for some

$$\xi' < \xi + \theta(\xi-\xi_0);$$

we then take

$$\left(\lim_{\xi\to\xi_0}\right) a(\xi) = \left(\lim_{\xi\to\xi_0}\right) g_{\xi_0}(\xi)$$

to have all, and only, the values belonging to

$$\boxed{g_{\xi_0}(\xi)}$$

for all ξ.

Now as

$$g_{\xi_0}(\xi_0+\delta) > g_{\xi_0}(\xi_0+\delta') \quad \text{for all} \quad \delta' \leqq \delta,$$

we see at once, by Theorem VIII, that the above definition of

$$\left(\lim_{\xi\to\xi_0}\right) g_{\xi_0}(\xi)$$

coincides precisely with that furnished by Theorem VIII for

$$\left(\lim_{m\to\infty}\right) g_{\xi_0}\left(\xi_0+\frac{1}{m}\right).$$

Let us extend the definition of the many-valued function $a(\xi)$ of the numerical variable ξ by defining

for any quantity x,

$$a(x)$$

to have *all, and only, the values each of which belongs to* $a(\xi)$ *for some value* ξ *of* x.

Then our function

$$g_{\xi_0}(\xi)$$

is precisely

$$a(\xi_0+\theta(\xi-\xi_0)),$$

where θ is our symmetrical sub-unit.

In fine, we therefore obtain our definition of the *finite limit in a continuous passage to the limit with respect to a numerical variable* ξ in the form

$$\left(\lim_{\xi\to\xi_0}\right) a(\xi) \equiv \left(\lim_{m\to\infty}\right) a\left(\xi_0+\frac{\theta}{m}\right).^{12)}$$

We notice that the many-valued function $a(x)$ of the many-valued variable x defined as above from $a(\xi)$, has the special property that

$$a(x') < a(x)$$

whenever

$$x' < x.$$

Functions with this property will be described as *contractive.*

If, for any *contractive many-valued function of a many-valued variable x*, we define, for each x_0, the auxiliary function

$$g_{x_0}(x)$$

as having (only) each value that belongs to

$$a(x')$$

for *some*

$$x' < x_0+\theta(x-x_0);$$

we have (since $a(x)$ is contractive),

$$g_{x_0}(x) = a(x_0+\theta(x-x_0)).$$

We then take

$$\left(\lim_{x\to x_0}\right) a(x) = \left(\lim_{x\to x_0}\right) g_{x_0}(x)$$

to have all, and only, the values belonging to $g_{x_0}(x)$ for every x, which are seen to be the values of

$$\left(\lim_{m\to\infty}\right) g_{x_0}\left(x_0+\frac{1}{m}\right).$$

We thus obtain (as definition of the left-hand side): *when* $a(x)$ *is a contractive function of* x

$$\left(\lim_{x\to x_0}\right) a(x) = \left(\lim_{m\to\infty}\right) a\left(x_0+\frac{\theta}{m}\right).$$

---

12) This limit includes $a(\xi_0)$. To obtain the more usual definition, replace θ by the quantity θ' having the same values except zero.

As a particular case, note that

$$\left(\lim_{x\to x_0}\right) x = \boxed{x_0}.$$

For unrestricted many-valued functions of x, or of other arguments (among which figure the variable point or set of points in n dimensions), in their full range or in restricted ranges, the same principle of course applies, although the formulae become more cumbersome, and new notations, hence to some extent new ideas, have to be devised for them.

In the case of a function

$$a(P)$$

of a variable point in n dimensions,

$$\left(\lim_{P\to P_0}\right) a(P)$$

reduces to an $n$ple finite limit.

Another interesting case is that in which the argument is a "subdivision", either in the Riemann or in the Lebesgue sense, of a given range of points, and we are dealing with limiting processes such as occur in Riemann and Young-Lebesgue integration. It may be described typically by saying that the argument is some object $\mathcal{P}$ with which is associated a specific quantity d, which we shall for definiteness call its *norm*, and the required limit is

$$\left(\lim_{d\to 0}\right) a(\mathcal{P}).$$

This is then defined by forming the contractive function of x

$$g(x)$$

having (exclusively) every value belonging to $a(\mathcal{P})$ for *some* $\mathcal{P}$ *of norm* $d < x\theta$, and equating the required limit to

$$\left(\lim_{x\to 0}\right) g(x) = \left(\lim_{m\to\infty}\right) g\left(\frac{\theta}{m}\right).$$

(Eingegangen am 23. 5. 1930.)

Verknüpfung einiger Rechenproben von R. Mehmke für das systematische Eliminieren bei linearen Gleichungssystemen mit bekannten Sätzen der Determinantentheorie.

Von

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1. Im 2. Hefte, S. 300–318 des vorangehenden 103. Bandes (1930) der Mathematischen Annalen hat R. Mehmke eine Arbeit „Praktische Lösung der Grundaufgaben über Determinanten, Matrizen und lineare Transformationen (Beiträge zur praktischen Analysis, II.)“ veröffentlicht. Er hebt in ihr gewisse für das praktische Rechnen wertvolle Proben hervor, die sich beim üblichen „gewöhnlichen“ Eliminieren und beim „beschleunigten“ Eliminieren von Unbekannten in linearen Gleichungssystemen ergeben und die nach seinen „Erkundigungen bei hervorragenden Sachkennern ... sonderbarerweise nicht bekannt zu sein scheinen“[1]). Die Beweise erbringt er durch Heranziehung der Rechnung mit „Extensen“ (Punkt- und Vektorenrechnung). Ich möchte hier zeigen, daß die Proben mit geläufigen Sätzen der Determinantentheorie identisch sind, die natürlich ihrerseits wieder mit jenen Tatsachen aus der Punkt- und Vektorenrechnung zusammenhängen. Und zwar handelt es sich um das Theorem von Sylvester[2]), nach dem die aus den $(h+1)$-reihigen Superdeterminanten $(1 \leqq h < n)$ von

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1h} \\ a_{21} & a_{22} & \cdots & a_{2h} \\ \cdot & \cdot & \cdots & \cdot \\ a_{h1} & a_{h2} & \cdots & a_{hh} \end{vmatrix}$$

[1]) Ein freilich nicht als Probe gewerteter Sonderfall findet sich z. B. bei P. B. Fischer, Determinanten, Sammlung Göschen 402, S. 6.

[2]) Vgl. G. Kowalewski, Einführung in die Determinantentheorie, Leipzig (Veit & Co.) 1909, § 41, S. 83–88, § 44, S. 99–102, § 45, S. 102–103.